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Geometry of PDE's. IV: Navier–Stokes equation and integral bordism groups

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Abstract

Following our previous results on this subject [R.P. Agarwal, A. Prástaro, Geometry of PDE's. III(I): Webs on PDE's and integral bordism groups. The general theory, Adv. Math. Sci. Appl. 17 (2007) 239–266; R.P. Agarwal, A. Prástaro, Geometry of PDE's. III(II): Webs on PDE's and integral bordism groups. Applications to Riemannian geometry PDE's, Adv. Math. Sci. Appl. 17 (2007) 267–285; A. Prástaro, Geometry of PDE's and Mechanics, World Scientific, Singapore, 1996; A. Prástaro, Quantum and integral (co)bordism in partial differential equations, Acta Appl. Math. (5) (3) (1998) 243–302; A. Prástaro, (Co)bordism groups in PDE's, Acta Appl. Math. 59 (2) (1999) 111–201; A. Prástaro, Quantized Partial Differential Equations, World Scientific Publishing Co, Singapore, 2004, 500 pp.; A. Prástaro, Geometry of PDE's. I: Integral bordism groups in PDE's, J. Math. Anal. Appl. 319 (2006) 547–566; A. Prástaro, Geometry of PDE's. II: Variational PDE's and integral bordism groups, J. Math. Anal. Appl. 321 (2006) 930–948; A. Prástaro, Th.M. Rassias, Ulam stability in geometry of PDE's, Nonlinear Funct. Anal. Appl. 8 (2) (2003) 259–278; I. Stakgold, Boundary Value Problems of Mathematical Physics, I, The MacMillan Company, New York, 1967; I. Stakgold, Boundary Value Problems of Mathematical Physics, II, Collier–MacMillan, Canada, Ltd, Toronto, Ontario, 1968], integral bordism groups of the Navier–Stokes equation are calculated for smooth, singular and weak solutions, respectively. Then a characterization of global solutions is made on this ground. Enough conditions to assure existence of global smooth solutions are given and related to nullity of *integral characteristic numbers* of the boundaries. Stability of global solutions are related to some characteristic numbers of the space-like Cauchy data. Global solutions of variational problems constrained by (*NS*) are classified by means of suitable integral bordism groups too.

Keywords: Navier–Stokes equation; Bordism groups; Global solutions; Instability; Variational calculus constrained by the Navier–Stokes equation

1. Preliminaries

This work is the fourth part of a series devoted to the study of PDE's geometry, by emphasizing the role played by (co)bordism techniques of the algebraic topology, suitably modified for PDE's, in order to characterize global solutions [1,14]. In this fourth part we specialize our theory on the Navier–Stokes equation. This is surely a master

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PDE to study and where apply any new theory of PDE's.[1] We will follow the same lines of some our previous works on this equation, adding some new results on the ground of the previous three parts issues.

The main results are the following. Theorem 2.2: The Cauchy problem is related to the symmetry group of the Navier–Stokes equation, showing as many solutions correspond to a same 3-dimensional space-like Cauchy integral manifold, having different space–time flows (*web-chaotic behaviour of the Navier–Stokes equation*). Theorems 2.5 and 2.6: A new characterization of integral bordism groups of the Navier–Stokes equation is made emphasizing the role played by singular solutions, weak solutions and their relations with smooth solutions. Theorems 2.8 and 2.9: Characterization of variational problems global solutions, constrained by the Navier–Stokes equation, by means of suitable bordism groups (*variational bordism groups*). Theorems 3.1, 3.2, 3.5: A new criterion on the full dynamic characterization of global solutions of the Navier–Stokes equation. Theorem 3.7: Existence of smooth global solutions, without singular points in its characteristic flow, for a large class of boundary value problems. With these last theorems we are able to identify smooth solutions which exhibit finite-time instability.

2. Integral bordism groups in (*NS*)

In this paper we consider the non-isothermal Navier–Stokes equation for incompressible Newtonian fluids, as a submanifold, $(NS) \subset JD^2(W)$, of the second-order jet-derivative space $JD^2(W)$ on the fiber bundle $\pi : W \equiv JD(M) \times_M T^0_0 M \times_M T^0_0 M \cong M \times \mathbf{I} \times \mathbb{R}^2 \to M$, over the affine 4-dimensional Galilean space–time M. (For notation see Refs. [12].) A section $s : M \to W$ of $\pi : W \to M$ is a triplet $s = (v, p, \theta)$, with v the velocity vector field, p the isotropic pressure field, and θ the temperature field. With respect to an inertial frame, (NS) is defined by the following equations: $\mathrm{div}(\rho v) = 0$, $\mathrm{div}[\rho v \otimes v - P] - \rho B = 0$, with $P = -pg + 2\chi\dot{e}$, $\rho\frac{\delta e}{\delta t} = \langle P, \dot{e}\rangle - \mathrm{div}(q)$, where $q = -\nu\,\mathrm{grad}(\theta)$. $\dot{e}$ denotes the infinitesimal strain, g is the vertical metric field of M, χ is the viscosity, ν the thermal conductivity, e the internal energy, and B the body volume force. We shall assume that ν, χ and $C_p \equiv (\partial\theta.e)$ are constant, and the body volume force conservative: $B = -\mathrm{grad}(f)$. The following theorem gives an important structure property of the Navier–Stokes equation.

Theorem 2.1. *(See [12].) Equation (NS) is an involutive but not formally integrable, and neither completely integrable, PDE of second-order on the fiber bundle $\pi : W \to M$. On the other hand, (NS) is a* universal regular *PDE, i.e., there exists into (NS) a sub-PDE $\widehat{(NS)}$, that is involutive formally integrable, and completely integrable, such that $\widehat{(NS)}_{+\infty} = (NS)_{+\infty}$. This means that all the solutions of (NS) are that of $\widehat{(NS)}$. (NS) is an affine fiber bundle over the affine submanifold $(C) \subset JD(W)$. (C) is a fiber bundle over W. Let us consider the following coordinate systems adapted to the frame ψ: M: (x^α), $0 \leqslant \alpha \leqslant 3$; W: $(x^\alpha, \dot{x}^i, p, \theta)$, $1 \leqslant i \leqslant 3$; $JD^2(W)$: $(x^\alpha, \dot{x}^i_\beta, p_\beta, \theta_\beta)$, $0 \leqslant \beta \leqslant 3$, $0 \leqslant |\beta| \leqslant 2$. Then, the expression of $\widehat{(NS)}$ in coordinates is given in Table* 1.[2]

Theorem 2.2 *(Cauchy problems and symmetries). For any admissible* 3*-dimensional space-like integral manifold there exist (global) solutions of (NS), but these are not unique.*[3]

Proof. We shall consider the following lemma.

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[1] For a panorama on the problems connected with this equation see Refs. [2,3,5,8,12,13,17,18].

[2] Here G^i_{jp} are the spatial components of the connection coefficients of the canonical connection on the Galilean space–time. The coefficients R^j_s can be written also in the form $R^j_s = \chi g^{pi}(\partial x_s.G^j_{ip})$. In fact taking into account that the (spatial) curvature R^j_{ips}, corresponding to the connection coefficients G^j_{is} is zero, we get $0 = R^j_{ips} = (\partial x_p.G^j_{is}) - (\partial x_s.G^j_{ip}) + G^j_{pq}G^q_{is} - G^q_{pi}G^j_{qs}$. It is important to emphasize that for the Navier–Stokes equation we do not necessitate to add further constraints in order to satisfy the principles of thermodynamics. In particular, any solution of the Navier–Stokes equation satisfies the second principle of the thermodynamics. This can be formulated saying that the body entropy production $\mathfrak{R}(s)$ is non-negative: $\mathfrak{R}(s) \geqslant 0$. For non-isothermal incompressible viscous fluids, like that described by equation (NS), $\mathfrak{R}(s)$ can be written in the following form: $\mathfrak{R}(s) = \frac{1}{\rho}[2\eta\dot{e}^{ij}\dot{e}_{ij} - q^i(\partial x_i.(\log\theta))] = \frac{1}{\rho}[2\eta\dot{e}^{ij}\dot{e}_{ij} + \nu(\mathrm{grad}\,\theta)^2]$, as $q^i = -\nu(\partial x_k.\theta)g^{ki}$ [11]. This numerical function is surely non-negative on all $JD(W)$. Therefore the second principle of the thermodynamics is surely satisfied from all solutions of (NS).

[3] This theorem definitely solves a question on the non-uniqueness of solutions for Cauchy problems in (NS) that was been conjectured just many years ago [8].

Table 1
Formally integrable Navier–Stokes equation: $\widehat{(NS)} \subset JD^2(W)$

(A): $F^0 \equiv \dot{x}^k G^j_{jk} + \dot{x}^i_s \delta^s_i = 0$; (B): $F^0_\alpha \equiv \dot{x}^k (\partial x_\alpha . G^j_{jk}) + \dot{x}^k_\alpha G^j_{jk} + \dot{x}^i_{s\alpha} \delta^s_i = 0$
(C): $F^j \equiv \dot{x}^s R^j_s + \dot{x}^s \dot{x}^i \rho G^j_{is} + \dot{x}^s \dot{x}^j_s \rho + \rho \dot{x}^j_0 + \dot{x}^k_s S^{js}_k + \dot{x}^j_{is} T^{is} + p_i g^{ij} + \rho (\partial x_i . f) g^{ij} = 0$
(D): $F^4 \equiv \theta_0 \rho C_p + \rho C_p \dot{x}^k \theta_k + \theta_{is} \overline{E}^{is} + \dot{x}^k \dot{x}^p W_{kp} + \dot{x}^k \dot{x}^s_p \overline{W}^p_{ks} + \dot{x}^k_i \dot{x}^s_p Y^{ip}_{ks} = 0$
$R^j_s \equiv \chi[(\partial x_p . G^j_{is}) + G^j_{pq} G^q_{is} - G^q_{pi} G^j_{qs}] g^{pi}$; $S^{js}_k \equiv \chi[2G^j_{ik} g^{si} - G^s_{qi} \delta^j_k g^{qi}]$
$T^{is} \equiv \chi g^{is} = T^{si}$; $\overline{E}^{is} \equiv -\nu g^{is} = \overline{E}^{si}$; $W_{kp} \equiv -\chi G^a_{jk} G^b_{sp} (g_{ba} g^{sj} + \delta^j_b \delta^s_a) = -2\chi G^s_{bk} G^b_{sp} = W_{pk}$
$\overline{W}^p_{ks} \equiv -2\chi[G^j_{ik} g_{js} g^{ip} + G^p_{sk}] = -4\chi G^p_{sk} = \overline{W}^p_{sk}$; $Y^{ip}_{ks} \equiv -\chi[g_{ks} g^{ip} + \delta^p_k \delta^i_s] = Y^{pi}_{sk}$;
Dual form of $\widehat{(NS)}$
$\delta \underline{v} = 0$; $(\partial x_\alpha . (\delta \underline{v})) = 0$; $\rho \frac{\delta v}{\delta t} = \chi \overline{\Delta \underline{v}} - \mathrm{grad}(p + \rho f)$; $\rho C_p \frac{\delta \theta}{\delta t} = \nu \Delta \theta + 2\chi \langle \dot{e}, \dot{e} \rangle$

$0 \leqslant \alpha \leqslant 3$; $1 \leqslant i, j, k, p, s \leqslant 3$. $\underline{v}$ is the vertical 1-form corresponding to v.
$\overline{\alpha}$ is the vertical vector field, corresponding to a vertical 1-form α.

Lemma 2.3 *(Infinitesimal symmetry algebra of $\widehat{(NS)}$). The infinitesimal symmetry algebra of $\widehat{(NS)}$, inducing on the space–time M admissible infinitesimal generator $\bar{\zeta} = \bar{\zeta}^\alpha \partial x_\alpha$ (i.e., with $\bar{\zeta}^0 = 1$ and $\mathrm{div}(\bar{\zeta}) = 0$), identifies space–time rigid flows. More preciesely we get that the infinitesimal symmetry algebra of $\widehat{(NS)}$ is generated by $\zeta = \bar{\zeta}^\alpha \partial x_\alpha + X^k \partial \dot{x}_k + Y \partial p + Z \partial \theta$, with $X^k = (\partial x_s . \bar{\zeta}^k) \dot{x}^s + (\partial x_0 . \bar{\zeta}^k) = (\partial x_\alpha . \bar{\zeta}^k) \dot{x}^\alpha$, $\dot{x}^0 = 1$, $Y = l(x^\alpha)$, $Z = \kappa \in \mathbb{R}$, $\bar{\zeta}^0 = 1$, $\mathrm{div}\, \bar{\zeta} = 0$, $L_{\bar{\zeta}} g^{ij} = 0$, $(\partial x_s . \bar{\zeta}^k) = \mathrm{const}$, $h^s_{/q} \equiv (\partial x_q . h^s) + G^s_{qi} h^i = 0$, $h^s \equiv (\partial x_0 . \bar{\zeta}^s)$, with $(\mathrm{grad}\, l)^e = -\rho (L_{\bar{\zeta}}\, \mathrm{grad}\, f)^e + \chi (\partial x_0 . \bar{\zeta}^s)[2G^e_{ik} G^k_{ps} - G^q_{pi} G^e_{qs} - (\partial x_s . G^e_{pi})] g^{pi} + \rho (\partial x_0 \partial x_0 . \bar{\zeta}^e)$. Furthermore, if $h^s = 0$ such flows are full rigid ones.*

Proof. We must impose the following conditions: $\zeta^{(2)} . F^0|_{\widehat{(NS)}} = 0$, $\zeta^{(2)} . F^0_\omega|_{\widehat{(NS)}} = 0$, $\zeta^{(2)} . F^e|_{\widehat{(NS)}} = 0$, $\zeta^{(2)} . F^4|_{\widehat{(NS)}} = 0$, where the symbols are as defined in Table 1. The conditions to stay on $\widehat{(NS)}$ are expressed by the equations that define $\widehat{(NS)}$ in solved form. By annulling the coefficients of the independent terms we get, after long, but direct, calculations the expression in coordinates for ζ given in the lemma. In the particular case that $h^s \equiv (\partial x_0 . \bar{\zeta}^s) = 0$, the symmetry vector field $\bar{\zeta}$ coincides with a full rigid one of the Galilean space–time M. □

Lemma 2.3 assures the existence of infinite solutions for any 3-dimensional integral space-like manifold contained into $\widehat{(NS)}$. In fact, any (regular) integral submanifold $N \subset \widehat{(NS)} \subset (NS) \subset J^2_4(W)$ of dimension 3 that satisfies the initial conditions and the transversality conditions, with respect to a time-like symmetry vector field $\zeta \in \mathfrak{s}(\mathfrak{C}_2)$ (where $\mathfrak{C}_2$ is the ideal encoding $\widehat{(NS)}$, and $\mathfrak{s}(\mathfrak{C}_2)$ denotes the space of infinitesimal symmetries of the ideal $\mathfrak{C}_2$), generates a (regular, smooth) solution of (NS) if N has no frozen singularities [9].[4] □

Example 2.4 *(Fourier heat equation).* Let us consider the Fourier equation of heat propagation $\rho C_p \theta_0 - \nu \theta_{ij} g^{ij} = 0$. This can be considered the non-isothermal Navier–Stokes equation with the additional constraints: $\dot{x}^k = \dot{x}^k_\alpha = \dot{x}^k_{\alpha\beta} = 0$. In particular, let us consider the classical simple form: $(F) \subset JD^2(W)$: $F \equiv \theta_t - a\theta_{xx} = 0$, that is we consider a 2-dimensional space–time M with coordinates (t, x), and $a \equiv \nu / \rho C_p$. Let us consider the following Cauchy problem: $\theta(0, x) = f(x)$ $(x > 0)$; $\theta(t, 0) = 0$, $t > 0$. This problem has the following unique regular solution [11,16]: $\theta(t, x) = \frac{2}{\pi} \int_{[0,\infty)} e^{-\mu^2 a t} \sin(\mu x)\, dx \int_{[0,\infty)} f(\xi) \sin(\mu \xi)\, d\xi$. This solution has the following characteristic vector field: $\zeta = \partial t + [\int_{[0,\infty)} e^{-(\mu^2 a) t} C(\mu) \sin(\mu x)\, d\mu] \partial \theta$, where the $C(\mu)$ is given by $C(\mu) = \frac{2}{\pi} \int_{[0,\infty)} f(\xi) \sin(\mu \xi)\, d\xi$. This can be obtained as symmetry time-like vector field of (F). The corresponding space–time flow-lines are straight-lines parallel to the time axis. Furthermore the corresponding heat vector flow is the following: $q = -\nu [\int_{[0,\infty)} e^{-(\mu^2 a) t} C(\mu) \mu \cos(\mu x)\, d\mu] \partial x$, that also goes to zero as the time goes to infinity. Note,

[4] Let us also emphasize that as (NS) and $\widehat{(NS)}$ are invariant for time-translations, generated by the vector field ∂x_0 on (NS), it follows that for any space-like admissible integral 3-dimensional manifold $N \subset \widehat{(NS)}$, there exists a canonical solution that represents the dynamical propagation of N with respect to the inertial frame considered and having zero relative velocity. The existence of this "canonical solution" does not justify to forget all the other ones.

also, that the body entropy productions of this solution is the following positive functions of (t,x): $R(s)=\frac{\nu}{\rho}(\frac{d\theta}{dx})^2=\frac{\nu}{\rho}(\int_{[0,\infty)}e^{-(\mu^2 a)t}C(\mu)\mu\cos(\mu x)\,d\mu)^2$. By concluding, the Fourier heat propagation equation has solutions, physically admissible in the framework of the classical non-relativistic mechanics, representing space–time propagation of the initial data with finite time-like velocity.

In order to pass from local existence theorems to global ones, it is necessary to consider the integral bordism group in dimension 3 of (*NS*). We shall adopt the same definitions and notations introduced in the previous three parts [1,14].

Theorem 2.5 *(Integral bordism groups in* (*NS*)*). One has the following isomorphisms*: $\bar{\Omega}_p^{(NS)}\cong\bar{H}_p((NS);\mathbb{R})\cong\Omega_{p,w}^{(NS)}\cong\Omega_{p,s}^{(NS)}\cong\bigoplus_{r+r'=p}H_r(M;\mathbb{Z}_2)\otimes_{\mathbb{Z}_2}\Omega_{r'}\cong\Omega_p$, $p\in\{0,1,2,3\}$, *where* Ω_p *is the bordism group for p-dimensional smooth closed manifolds. Therefore, the integral bordism group for* (*weak*) *singular solutions is* $\Omega_{3,w}^{(NS)}\cong\Omega_{3,s}^{(NS)}\cong\Omega_3^{(NS)}/K_{3,s}^{(NS)}\cong\Omega_3=0$, *where* $K_{3,s}^{(NS)}$ *is the subgroup of* $\Omega_3^{(NS)}$, *such that* $[N]\in K_{3,s}^{(NS)}$ *iff* $N=\partial V$, *with V a singular solution of* (*NS*). *For smooth solutions one has*: $\Omega_3^{(NS)}\cong\Omega_3^{(NS)+\infty}$. *If are considered admissible all the* 3*-dimensional closed smooth integral manifolds of* (*NS*) *that are just admissible in the previous sense and that have zero all the integral characteristic numbers* (*see parts* I–III) (full admissibility hypothesis)*, then* $\Omega_3^{(NS)}=0$.

Proof. As (*NS*) is not completely integrable and the set of all possible initial conditions admitting solutions coincides with $(\widehat{NS})$, we can substitute (*NS*) with $(\widehat{NS})$. Then taking into account that $(\widehat{NS})$ is involutive, formally integrable and completely integrable, with $\dim(\widehat{NS})=70>2\times4+1=9$, we can apply Theorem 2.15 given in part I, and Theorems 2.4 and 2.11 in part III(I), to obtain the integral bordism groups of $(\widehat{NS})$ and, hence, also of (*NS*). Note that one has the following short exact sequence: $0\to K_{p,s}^{(NS)}\to\Omega_p^{(NS)}\to\Omega_{p,s}^{(NS)}\to0$, and the following isomorphism $\Omega_{p,s}^{(NS)}\cong\underline{\Omega}_p(W)$, where $\underline{\Omega}_p(W)$ is the p-bordism group of W. Then considering also that $\pi:W\to M$ is an affine bundle we get $\Omega_{p,s}^{(\widehat{NS})}\cong\underline{\Omega}_p(W)\cong\underline{\Omega}_p(M)\cong\bigoplus_{r+r'=p}H_r(M;\mathbb{Z}_2)\otimes_{\mathbb{Z}_2}\Omega_{r'}\cong\Omega_p$, for $p\in\{0,1,2,3\}$. In particular, one has: $\Omega_{p,s}^{(NS)}\cong0$, for $p=1,3$, and $\Omega_{p,s}^{(NS)}\cong\mathbb{Z}_2$, for $p=0,2$. Such integral bordism groups refer to solutions of (*NS*) as formalized in part III(I), hence include (weak) singular solutions in the sense there defined. In order to be sure that a solutions $V\subset(NS)$ is smooth one, it is enough to assure that all the integral characteristic numbers of the boundary, $N=\partial V$, are zero. Let us recall that the involved integral characteristic numbers are all the evaluations of (differential) conservation laws of $(\widehat{NS})$ on N, i.e., of all the differential 3-forms belonging to $\mathfrak{C}_{\mathrm{cons}}((\widehat{NS}))^3\equiv\frac{\Omega^3((\widehat{NS})_{+\infty})\cap d^{-1}C\Omega^4((\widehat{NS})_{+\infty})}{C\Omega^3((\widehat{NS})_{+\infty})\oplus d\Omega^2((\widehat{NS})_{+\infty})}$, where $\Omega^q((NS)_{+\infty})$, $q=0,1,2,\dots$, is the space of differential q-forms on $(\widehat{NS})_{+\infty}$, $C\Omega^q((\widehat{NS})_{+\infty})$ is the space of all Cartan q-forms on $(\widehat{NS})_{+\infty}$, $q=1,2,\dots$, and $C\Omega^0((\widehat{NS})_{+\infty})\equiv0$, $C\Omega^q((\widehat{NS})_{+\infty})\equiv\Omega^q((\widehat{NS})_{+\infty})$, for $q>4$, $\Omega^{-1}((\widehat{NS})_{+\infty})=0$. $C\Omega^q((\widehat{NS})_{+\infty})\equiv\{\beta\in\Omega^q((\widehat{NS})_{+\infty})\mid\beta(\zeta_1,\dots,\zeta_q)(p)=0,\ \zeta_i(p)\in(\hat{\mathbf{E}}_\infty)_p,\ \forall p\in(\widehat{NS})_{+\infty}\}$. Here $\hat{\mathbf{E}}_\infty$ is the Cartan distribution on $(\widehat{NS})_{+\infty}$. $\square$

Theorem 2.6 *(p-Conservation laws of* $(\widehat{NS})$*). The p-conservation laws,*

$$[\omega]\in\mathfrak{C}_{\mathrm{cons}}((\widehat{NS}))^p\equiv\frac{\Omega^p((\widehat{NS})_{+\infty})\cap d^{-1}C\Omega^{p+1}((\widehat{NS})_{+\infty})}{C\Omega^p((\widehat{NS})_{+\infty})\oplus d\Omega^{p-1}((\widehat{NS})_{+\infty})},\quad p=0,1,2,3,$$

of $(\widehat{NS})$ *are represented, in adapted coordinates, by the following differential forms.* $(p=0)$: $f\in C^\infty(\widehat{NS})_{+\infty}$, *with* $f=c\in\mathbb{R}$, *i.e.,* $\mathfrak{Cons}((\widehat{NS}))^0=\mathbb{R}$. $(p=1,2)$: $\mathfrak{Cons}((\widehat{NS}))^p=0$. $(p=3)$: $\omega=\omega_{012}\,dx^0\wedge dx^1\wedge dx^2+\omega_{013}\,dx^0\wedge dx^1\wedge dx^3+\omega_{023}\,dx^0\wedge dx^2\wedge dx^3$, *where* $\omega_{012}=\omega_{012}(x^\alpha,\sqrt{|g|}\dot{x}_0^3)$, $\omega_{013}=\omega_{013}(x^\alpha,\sqrt{|g|}\dot{x}_0^2)$, $\omega_{023}=\omega_{023}(x^\alpha,\sqrt{|g|}\dot{x}_0^1)$, *are smooth functions, such that*: $-(\partial x_3.\omega_{012})+(\partial x_2.\omega_{013})-(\partial x_1.\omega_{023})=0$.

Proof. In fact, any p-conservation law of $(\widehat{NS})$ can be written in adapted coordinates, as follows: $\omega=\omega_{\mu_1\cdots\mu_p}\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}$, with $\omega_{\mu_1\cdots\mu_p}$ smooth functions on $(\widehat{NS})_{+\infty}$, such that $(\partial_{[\mu}.\omega_{\mu_1\cdots\mu_p]})=0$, where $\partial_\mu=\partial x_\mu+\sum_{|\alpha|\geqslant0}y^j_{\mu\alpha}\partial y^\alpha_j$, is the natural basis for $\widehat{\mathbf{E}}_\infty\subset T(\widehat{NS})_{+\infty}$, i.e., $y^j_{\mu\alpha}$ are vertical coordinate functions on $JD^\infty(W)$, that satisfy the equations for $(\widehat{NS})_{+\infty}$. Then after some long but direct calculations we obtain the theorem. $\square$

Example 2.7. The conservation law, given in Ref. [12], $\omega=\sqrt{|g|}[\dot{x}_0^3\,dx^0\wedge dx^1\wedge dx^2-\dot{x}_0^2\,dx^0\wedge dx^1\wedge dx^3+\dot{x}_0^1\,dx^0\wedge dx^2\wedge dx^3]$, is just of the type considered in Theorem 2.6, with $\omega_{023}=\sqrt{|g|}\dot{x}_0^1$, $\omega_{013}=\sqrt{|g|}\dot{x}_0^2$, $\omega_{012}=\sqrt{|g|}\dot{x}_0^3$. Note that by using non-slip hypothesis on the, at the rest, boundary ∂B of a continuum system, one has that $\int_N \omega|_N=0$, where N is the integral manifold $N=B_1\cup_{\partial B_1} X\cup_{\partial B_2} B_2$, where $X\subset\widehat{(NS)}$ is the time-like integral manifold of $\widehat{(NS)}$ obtained as a propagation of ∂B, i.e., $N=\partial V$, with V a solution of $\widehat{(NS)}$, having B_1 as an initial Cauchy hypersurface and B_2 as a final Cauchy hypersurface, respectively.

Let us consider, here, as further applications of part II, and above results, some variational problems constrained by (NS). We shall see that our theory allows us to characterize global solutions for such problems.

Theorem 2.8 *(Existence of global solutions with extrema in the kinetic energy dissipation rate). For any (smooth) boundary condition there exist (smooth) global solutions with extrema in the kinetic energy dissipation rate. Furthermore, such solutions are smooth if the boundary are smooth and have zero all the integral characteristic numbers (i.e. they are fully admissible).*

Proof. Let us consider the following 4-variational system $(X=(NS),\mathfrak{C}(X),\theta)$, where $\mathfrak{C}(X)$ is the contact ideal of (NS). Furthermore, θ is a Lagrangian density $\theta=L\sqrt{|g|}\,dx^0\wedge dx^1\wedge dx^2\wedge dx^3$ with $L:(NS)\to\mathbb{R}$, (g_{ij}) the vertical metric on M, and $|g|\equiv\det(g_{ij})$. As (NS) is a universally regular with the subequation $\widehat{(NS)}\subset(NS)\subset JD^2(W)$ involutive, formally integrable (and completely integrable), the equivalence class $[\widehat{(NS)}]\in\mathfrak{D}_f$ has $\widehat{(NS)}_{+\infty}\subset JD^\infty(W)$ as universal PDE (see notation in part II). Then, the Euler–Lagrange equation of the variational system can be identified with a sub-equation $E_\infty\subset\widehat{(NS)}_{+\infty}\subset JD^\infty(W)$. Furthermore $H^{4+1}_{\mathrm{var}}((NS)_{+\infty})=H^{4+1}_{\mathrm{var}}(\widehat{(NS)}_{+\infty})=H^5_{\mathrm{var}}(\widehat{(NS)})=H^5_{\mathrm{var}}(NS)=0$ as $\widehat{(NS)}$ and (NS) are affine subbundles of $JD^2(W)\to JD(W)$, and (NS) is universally regular. Hence (NS) is also wholly variational. (This is equivalent to say that $H^q_{\mathrm{var}}((NS))\cong H^q_{\mathrm{var}}((NS)_{+\infty})\cong H^q(W)$, at least for $q=4+1$.) Therefore the inverse variational problems admit global solutions. An important example of variational problem constrained by the Navier–Stokes equation is to find solutions with the smallest possible energy dissipation rate for fixed boundary conditions. For such equations, by calculating their integral bordism groups, following the general methods introduced by us, we can prove the existence of global (smooth) solutions for any (smooth) boundary condition. More precisely, let us consider the following Lagrangian density on $\widehat{(NS)}$: $\theta=\sqrt{|g|}\chi(\overline{\Delta\underline{v}}\cdot v)\,dx^0\wedge dx^1\wedge dx^2\wedge dx^3\equiv\chi(\overline{\Delta\underline{v}}\cdot v)\,dx^0\wedge\eta\equiv L\,dx^0\wedge dx^1\wedge dx^2\wedge dx^3$. Its physical meaning is obtained by considering the equation for the kinetic energy density: $\frac{1}{2}\rho\frac{\delta v^2}{\delta t}=-\langle vd(p+\rho f),v\rangle+\chi(\overline{\Delta\underline{v}}\cdot v)$. This can be obtained, by scalar multiplication by v, the third equation of the dual form for $\widehat{(NS)}$ in Table 1. By integration on a time section $B_t\subset M$ we get: $\int_{B_t}\frac{1}{2}\rho\frac{\delta v^2}{\delta t}\eta=\int_{B_t}\chi(\overline{\Delta\underline{v}}\cdot v)\eta=-\chi\int_{B_t}((v_k)_{/i})_{/j}g^{ij}v^k\eta$. The term $\epsilon(t)\equiv\int_{B_t}\chi(\overline{\Delta\underline{v}}\cdot v)\eta$ is called *kinetic energy dissipation rate* (at the time t). Then the integral action corresponding to the above Lagrangian density represents the total kinetic energy dissipation. Therefore we can formulate the following constrained variational problem: To find solutions of $\widehat{(NS)}$ that have extrema in kinetic energy dissipation. We can solve this problem without using Lagrange multipliers. In fact, we can consider the variation problem defined by the Lagrangian L and use results given in part II. More precisely, the extrema are solutions of $\widehat{(NS)}$ such that (♠♠): $\int_A\langle\nu^I,\mathfrak{E}_I\rangle\,dx^0\wedge dx^1\wedge dx^2\wedge dx^3=0$, where (ν^I) are the vertical vector fields on W, defining the infinitesimal symmetry algebra of $\widehat{(NS)}$. Taking into account that L can be written in adapted coordinates as follows: $L=-\sqrt{|g|}\chi(\dot{x}^e_{ij}+A^{qe}_{pji}\dot{x}^p_q+B^e_{hij}\dot{x}^h)g_{et}g^{ji}\dot{x}^t$, where $A^{qe}_{pji}\equiv G^e_{ih}\delta^h_p\delta^q_j+G^e_{sj}\delta^s_p\delta^q_i-G^s_{ij}\delta^e_p\delta^q_s$, $B^e_{hij}\equiv(\partial x_j.G^e_{ih})+G^e_{sj}G^s_{ih}-G^s_{ij}G^e_{sh}$, we get for $\mathfrak{E}_I$ the following expressions: $\mathfrak{E}_k\equiv(\partial\dot{x}_k.L)-(\partial_\alpha(\partial\dot{x}^\alpha_k.L))+(\partial_\alpha\partial_\beta(\partial\dot{x}^{\alpha\beta}_k.L))=\dot{x}^hC_{hk}+\dot{x}^p_qD^q_{kp}+\dot{x}^e_{ij}E^{ij}_{ek}=0$, $\mathfrak{E}_p\equiv(\partial p.L)-(\partial_\alpha(\partial p^\alpha_k.L))+(\partial_\alpha\partial_\beta(\partial p^{\alpha\beta}_k.L))\equiv0$, $\mathfrak{E}_\theta\equiv(\partial\theta.L)-(\partial_\alpha(\partial\theta^\alpha_k.L))+(\partial_\alpha\partial_\beta(\partial\theta^{\alpha\beta}_k.L))\equiv0$. For (ν^I), by using Lemma 2.2, we have: $(\nu^I)=(\nu^k,\nu^p,\nu^\theta)=(a^k_sb^s,l,\kappa)$, where a^k_s, l and κ are arbitrary constants and (b^s), $k,s=1,2,3$, are arbitrary numerical functions on M. (Here we assume the body force with gravitational potential and $h^s=0$ in Lemma 2.3.) Therefore extrema are solutions of the following PDE: $E_2\subset JD(W)$: $\{\mathfrak{E}_k\equiv\dot{x}^hC_{hk}+\dot{x}^p_qD^q_{kp}+\dot{x}^e_{ij}E^{ij}_{ek}=0,\ 1\leqslant k\leqslant3;\ \widehat{(NS)}$: $\{F^0=0;\ F^0_\gamma=0;\ F^j=0;\ F^4=0\}\}$. This PDE is involutive formally integrable and completely integrable. This means that we can find local solutions of such a constrained variational problem for any initial condition of equation $E_2\subset JD^2(W)$. Let us now consider the characterization of global solutions of E_2 by means of its integral bordisms. We can apply Theorem 2.5 and see that the integral bordism groups of E_2 are the same as those of $\widehat{(NS)}$. This has as an important consequence: the existence of(smooth) global solutions with extrema in the kinetic energy dissipation rate, for any (smooth) boundary condition of equation $E_2 \subset J^2_4(W)$. In fact, one has $\Omega_{3,s}^{(E_\infty)_{\mathrm{var}}} \cong \Omega_{3,s}(\widehat{(NS)}_{+\infty}, (E_\infty)_{\mathrm{var}}) \cong \Omega_{3,s}^{\widehat{(NS)}_{+\infty}} \cong 0$. Furthermore in the hypothesis of full admissibility for the smooth boundary, one has also $\Omega_3^{(E_\infty)_{\mathrm{var}}} \cong \Omega_3(\widehat{(NS)}_{+\infty}, (E_\infty)_{\mathrm{var}}) \cong \Omega_3^{\widehat{(NS)}_{+\infty}} \cong 0$. Therefore, we get also the existence of smooth global solutions. □

Theorem 2.9 *(Solutions with extrema in the body entropy production). This problem does not admit* (*local*) *solution for any initial condition.*

Proof. Let us consider the following 4-variational system $(X = \widehat{(NS)}, \mathfrak{C}(X), \theta)$, where $\widehat{(NS)} \subset JD^2(W)$ is the non-isothermal Navier–Stokes equation for incompressible fluids on the Galilean space–time M, as before, and θ is the Lagrangian density $\theta = L\sqrt{|g|}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ with $L \equiv \frac{1}{\rho}\sqrt{|g|}[\eta(\dot{x}^j_r \dot{x}^r_j + \dot{x}^i_s \dot{x}^a_j g^{sj} g_{ai} + 3G^s_{ip}\dot{x}^p \dot{x}^i_s 2G^i_{sp} G^s_{ic}\dot{x}^p \dot{x}^i) + \frac{\nu}{\theta}\theta_j \theta_i g^{ji}] : \widehat{(NS)} \to \mathbb{R}$ the body entropy production. (See in Section 1.) Then taking into account that, similarly to the previous theorem, now one has: $\mathfrak{E}_p \equiv 0$, $\mathfrak{E}_\theta \equiv 0$, $\mathfrak{E}_k = \dot{x}^p_{j\alpha} A^{j\alpha}_{pk} + \dot{x}^\beta_\gamma B^\gamma_{pk} + \dot{x}^p C_{pk}$, we get that extrema for this constrained variational problem are solutions of the following PDE: $E_2 \subset JD^2(W)$: $\{\mathfrak{E}_k = \dot{x}^p_{j\alpha} A^{j\alpha}_{pk} + \dot{x}^\beta_\gamma B^\gamma_{pk} + \dot{x}^p C_{pk} = 0;\ \widehat{(NS)}\ \{F^0 = 0, F^0_\gamma = 0, F^5_j = 0, F^4 = 0\}\}$, where $A^{j\alpha}_{pk} \equiv -\frac{2\eta}{\rho}\sqrt{|g|}(G\delta^\alpha_p \delta^j_k + \delta^\alpha_s g^{sj} g_{pk})$, $B^\gamma_{pk} \equiv -\frac{2\eta}{\rho}[\sqrt{|g|}(\partial x_s.(g^{sj} g_{pk}))\delta^\gamma_j + (\partial x_\alpha.\sqrt{|g|})(\delta^\alpha_p \delta^\gamma_k + \delta^\alpha_s \delta^\gamma_j g^{sj} g_{pk})]$, $C_{pk} \equiv \frac{\eta}{\rho}\sqrt{|g|}[2(G^i_{sk} G^s_{pi} + G^i_{sp} G^s_{ik}) - 3(\partial x_s.G^s_{kp})] - \frac{3\eta}{\rho}(\partial x_s.\sqrt{|g|})G^s_{kp}$, $\bar{A}^{ji} \equiv -\frac{2\nu}{\rho}\sqrt{|g|}g^{ij}$, $\bar{B}^i \equiv -\frac{2\nu}{\rho}\frac{1}{\theta}(\partial x_j.(g^{ij}\sqrt{|g|})) = \frac{1}{\theta}(\partial x_j.\bar{A}^{ij})$. This equation is not involutive formally integrable and completely integrable. In fact, the canonical mapping $(E_2)_{+1} \to E_2$ cannot be surjective. This means that the this problem does not admit (local) solution for any initial condition of equation E_2. So to solve this problem it is necessary to add some constraint in order to obtain a formally integrable PDE $\widehat{E_{2+r}} \subset JD^{2+r}(W)$. We skip on details. □

3. Boundary value problems in (*NS*)

We consider, now, the question if to each smooth boundary value problem on (*NS*) one can associate a unique global (smooth) solution. (We define *global* (*smooth*) *solution* of (*NS*) any (smooth) solution of (*NS*) that is defined in any space-like region inside the boundary and for any $t \geqslant t_0$, where t_0 is an initial time.) We have just proved that for the Navier–Stokes equation we have not, in general, the uniqueness of solutions for Cauchy problems. Here we shall see that also the uniqueness of solutions for general boundary value problems is not assured. These facts are essentially related to the integral bordism groups of (*NS*), its symmetry properties, and (in)stability properties of solutions.

Theorem 3.1 *(Existence of global (smooth) solutions of* (*NS*)*). For a generic* (*smooth*) *boundary value problem contained into* $\widehat{(NS)} \subset (NS)$ *exist global* (*smooth*) *solutions of* (*NS*)*. If the boundary is smooth and has zero all the integral characteristic numbers, then there exist smooth global solutions too.*

Proof. A boundary value problem for (*NS*) can be directly described in the manifold $(NS) \subset JD^2(W) \subset J^2_4(W)$ by requiring that a 3-dimensional compact space-like (for some $t = t_0$), admissible integral manifold $B \subset \widehat{(NS)} \subset (NS)$ propagates in (*NS*) in such a way that the boundary ∂B describes a fixed 3-dimensional time-like integral manifold $Y \subset \widehat{(NS)} \subset (NS)$. We shall require that the boundary ∂B of B is orientable. Y is not, in general, a closed (smooth) manifold. However, we can solder Y with two other compact 3-dimensional integral manifolds X_i, $i = 1, 2$, in such a way that the result is a closed 3-dimensional (smooth) integral manifold $Z \subset \widehat{(NS)} \subset (NS)$. More precisely, we can take $X_1 \equiv B$, so that $\widetilde{Z} \equiv X_1 \cup_{\partial B} Y$ is a 3-dimensional compact integral manifold such that $\partial \widetilde{Z} \equiv C$ is a 2-dimensional space-like integral manifold. We can assume that C is an orientable manifold. Then from Theorem 2.5 it follows that $\partial X_2 = C$, for some space-like compact 3-dimensional integral manifold $X_2 \subset \widehat{(NS)}$. Set $Z \equiv \widetilde{Z} \cup_C X_2$. Therefore, one has $Z = X_1 \cup_{\partial B} Y \cup_C X_2$. Then, again from Theorem 2.5 it follows that there exists a 4-dimensional integral (smooth) manifold $V \subset \widehat{(NS)} \subset (NS)$ such that $\partial V = Z$. It follows that the integral manifold V is a solution of our boundary value problem between the times t_0 and t_1, where t_0 and t_1 are the times corresponding to the boundaries where are soldered X_i, $i = 1, 2$ to Y. Now, this process can be extended for any $t_2 > t_1$. So we are able to find (smooth) solutions for any $t > t_0$. Therefore we are able to find global (smooth) solutions. Remark that such global solutions, obtained by gluing smooth manifolds together, are 4-dimensional topological manifolds that are, in general, piecewise smooth only. However, in order to assure the smoothness of the global solutions so built it is enough to develop such constructions in the infinity prolongation $\widehat{(NS)}_{+\infty}$ of $\widehat{(NS)}$, when all the integral characteristic numbers of the boundary are zero. □

Theorem 3.2. *Boundary conditions do not assure, in general, the uniqueness of solutions.*[5]

Proof. Let us proceed in some fundamental steps. (For more details see also Ref. [12].) The integral bordism group $\Omega_{3,s}^{(NS)} = 0$ and the symbols of $\widehat{(NS)}$ and $\widehat{(NS)}_{+1}$ are non-zero. This is enough to assure existence of many solutions that have the same time-like 3-dimensional boundary. Furthermore, regular (smooth) solutions with respect to their projections on W, do not necessitate to be regular with respect to their projections on M. For the Navier–Stokes equation with viscosity $\chi \neq 0$, the Reynolds, Frank and Brindeman numbers cannot, in general, identify conservation laws and neither 1-conservation laws. However, these adimensional numbers, identified with characteristic numbers of space-like domains of the fluids, are conserved into a solution $V \subset (NS)$, i.e., are characteristic numbers of V, iff the mean's shear strain is zero: $\int_{\underline{B}_t} u^i u^j \dot{e}_{ij} \eta = 0$, where $\underline{B}_t \subset M$ is any fixed 3-dimensional space-like domain corresponding to the 3-dimensional integral space-like manifold $N_t \subset V_t$, $V_t \cap (NS)_t$, $t \in T$, u^i are the space-like components of the distribution of velocity corresponding to the solution V. For each solution $V \subset (NS)$ define the following characteristic function (*fundamental global rate*): $\dot{\mathfrak{f}} = \dot{\mathfrak{f}}(t) \equiv \frac{d}{d\lambda}(R_e^2(\zeta|B_{t+\lambda}) - R_e^2(\zeta|B_t))|_{\lambda=0}$. The fundamental global rate is proportional to the mean's shear rate: $\dot{\mathfrak{f}}(t) = (\frac{\rho L}{\chi})^2 \frac{2}{\mathrm{vol}(\underline{B}_t)} \int_{\underline{B}_t} u^i u^j \dot{e}_{ij} \eta$. Therefore, the Reynolds number $R_e(\zeta|B_t)$ (respectively the Frank number $F_r(\zeta|B_t)$, the Brindeman number $B_r(\zeta|B_t)$), is a characteristic number of the solution V, i.e., it is conserved, iff $\int_{\underline{B}_t} u^i u^j \dot{e}_{ij} \eta = 0$. (In such a case we write $R_e(V)$ (respectively $F_r(V)$, respectively $B_r(V)$), to denote the characteristic number of V, just associated to the Reynolds number (respectively Frank number, respectively Brindeman number).) In the set $\mathfrak{S}_{ol}(NS)$ of solutions of the Navier–Stokes equation (NS), there are also ones where are present different domains, some corresponding to stable flows and other to unstable flows. Those solutions V such that $\dot{\mathfrak{f}} = 0$, and with the characteristic number $R_e(V)$ lower than a *critical value*, are stable solutions (assumed that the temperature and pressure perturbations are neglectable). More precisely, let us consider the linearized $\widehat{\widehat{(NS)}_s}$ of $\widehat{(NS)}$ at the solution s. This is an involutive, formally integrable and completely integrable linear PDE. Therefore, to any initial condition of $\widehat{\widehat{(NS)}_s}$, i.e., to any admissible perturbation, at some point, of the solution s, will correspond a local solution, i.e., a local perturbation of the solution s. Furthermore, the Cauchy problem for $\widehat{\widehat{(NS)}_s}$ admits regular solutions for any space-like regular 3-dimensional data. In fact, the pseudogroup G of symmetries of $\widehat{(NS)}$ induces a pseudogroup G_s of symmetries of $\widehat{\widehat{(NS)}_s}$. More precisely one has that $G_s \subset G$ is identified by means of all (local) fiber bundle transformations $(\phi, \bar{\phi})$ of $\pi : W \to M$, belonging to G, such that $s' \equiv \phi^* s \equiv \phi \circ s \circ \bar{\phi}^{-1} = s$. Therefore $\phi^{(2)}(\widehat{\widehat{(NS)}_s}) \subset \widehat{\widehat{(NS)}_{s'}}$ iff $s = s' = \phi^* s$. Now, a Cauchy problem for $\widehat{\widehat{(NS)}_s}$ cannot be solved with respect to a space-like Cauchy data manifold, by using time-translations (like can be made, for example, for $\widehat{(NS)}$), since it should require that $\widehat{\widehat{(NS)}_s}$ is invariant for time translations. This happens only if s is time-translations invariant (i.e., s is a steady-state solution). However, in general, the Cauchy problem for $\widehat{\widehat{(NS)}_s}$ admits solutions for any space-like regular 3-dimensional data, $C_t \subset \widehat{\widehat{(NS)}_s}$, as we can use the flow ϕ_λ induced by the characteristic flow that solves the corresponding Cauchy problem for s. In fact, for such a flow one has $\phi_\lambda^* s = s$. Furthermore, if $C_t = D^2 \nu_0(B_t)$, where $B_t \subset M$ is a compact 3-dimensional space-like submanifold of $M_t \equiv \tau^{-1}(t) \subset M$, and $\nu_0 : B_t \to s^* vTW$ is a section over B_t, then $\phi_\lambda^* \nu_0 = vT(\phi_\lambda) \circ \nu_0 \circ \bar{\phi}_\lambda^{-1} : M \to s^* vTW$ is a solution of $\widehat{\widehat{(NS)}_s}$. In fact, $D^2(\phi_\lambda^* \nu_0)(p) = (JD^2(vT(\phi_\lambda) \circ D^2 \nu_0 \circ \phi_\lambda^{-1})(p) \in \widehat{\widehat{(NS)}_s}$, for any $p \in \bar{\phi}_\lambda(B_t)$, as $D^2 \nu_0(\phi_\lambda^{-1}(p)) \in C_t \subset \widehat{\widehat{(NS)}_s}$ and $JD^2(vT(\phi_\lambda))$ is a diffeomeorphism of $\widehat{\widehat{(NS)}_s}$. One has the following criterion of existence and uniqueness.

[5] The non-uniqueness of solutions, in general, for the Navier–Stokes equation, stated in this theorem, does not exclude that adding some further constraints one can obtain uniqueness of solutions. (See, e.g., Example 2.4, example in nota 8, and Refs. [2,16–18].)

Lemma 3.3. *(See H. Goldschmidt and D. Spencer [6].) The existence and unicity of solutions for linear Cauchy problems can be reconduced to a canonical mapping $\rho_* : H^{q,j}(E_k)_{\rho(x)} \to H^{q,j}(E_k)_{\rho,x}$, $\forall x \in X$, $q \geqslant k$, $j \geqslant 0$, between Spencer cohomology spaces, where $E_k \subset JD^k(W)$ is a linear PDE on a vector fiber bundle $\pi : E \to M$, and $\rho : X \to M$ is the embedding mapping of an s-dimensional submanifold $X \subset M$. Surjectivity of these maps corresponds to the existence theorems and injectivity to uniqueness. If g_{k+1} is involutive and X is non-characteristic for g_{k+1} one has the existence and uniqueness for the Cauchy problem, i.e., there exists a unique section s of E defined on a neighbourhood U of X in M such that $D^k s(x) \in E_k$, $\forall x \in U$ and $s|_X = s_0 \in C^\omega(E|_X)$, for some fixed s_0.*

Let us, now, apply above results to our involutive, formally integrable linear equation $\widehat{(NS)_s}$, by specializing the submanifold $X \subset M$ with a 3-dimensional space-like submanifold. Then, in such a case we can see that the maps $\rho_* : H^{q,0}(\widehat{(NS)_s})_{\rho(x)} \to H^{q,0}(\widehat{(NS)_s})_{\rho,x}$, $\forall x \in X$, $q \geqslant 2$, is surjective. In fact, we have just seen that $\widehat{(NS)_s}$ admits solutions for any space-like regular 3-dimensional data. On the other hand the uniqueness fails. In fact, one can easily see that such 3-dimensional space-like submanifolds do not satisfy the condition to be non-characteristic for the symbol $\widehat{g}_{2+1}[s]$ of the first prolongation of $\widehat{(NS)_s}$.[6] In fact, let us consider the submanifold identified by the equation $x^0 =$ const in the coordinates (x^α) on M, adapted to an inertial frame. Then the induced coordinates on TM and TX are $(x^\alpha, \dot{x}^\alpha)$ and $(x^k, \dot{x}^k)$, respectively. Therefore, on the normal bundle $TM|_X/TX$ the induced coordinates are $(x^k, \dot{x}^0)$, hence on the conormal bundle N one has $(x^k, \dot{x}_0)$. This means that a vector $\zeta \in S^3_0(N) \otimes E$ iff its representation in coordinates is $\zeta = \zeta^I_{000}(x^k)\, dx^0 \otimes dx^0 \otimes dx^0 \otimes \partial y_I$, where $(y^I) \equiv (\dot{x}^k, p, \theta)$ are vertical coordinates on E. Now, writing down in adapted coordinates the equation $\widehat{(NS)_s}$ and its first prolongation, we can see that the corresponding symbol $(\widehat{g_2[s]})_{+1} \subset S^3_0(N) \otimes E$ is characterized by vector fields $\zeta^I_{\alpha\beta\gamma}\, dx^\alpha \odot dx^\beta \odot dx^\gamma \otimes \partial y_I = \zeta^I_{\alpha\beta\gamma} \partial y_I^{\alpha\beta\gamma}$ such that the following equations are satisfied: $(\partial \dot{x}_k^{\beta p}.F^0_\alpha)_s \zeta^k_{\beta p\gamma} = 0$; $(\partial \dot{x}_k^{ir}.F^j)_s \zeta^k_{ir\gamma} = 0$; $(\partial \theta^{ir}.F^4)_s \zeta^{(\theta)}_{ir\gamma} = 0$. Therefore, $(S^3_0(N) \otimes E) \cap (\widehat{g_2[s]})_{+1} \neq 0$ as we have not restrictions on the components ζ^I_{000} of $\zeta^I_{\alpha\beta\gamma} \partial y_I^{\alpha\beta\gamma} \in (\widehat{g_2[s]})_{+1}$. Finally, in order to recognize the existence of global smooth solutions of $\widehat{(NS)_s}$ it is enough to calculate their integral bordism groups. In particular, by applying results of parts I and III(I), we see that $\Omega^{\widehat{(NS)_s}}_{3,s} \cong 0$. This is enough to state the existence of global (smooth) solutions of $\widehat{(NS)_s}$ for any admissible Cauchy data. If s is stationary the coefficients in the expression of $\widehat{(NS)_s}$ are functions on M independent on x^0. Then the general regular solution of $\widehat{(NS)_s}$ can be obtained as a linear combinations of solutions of the type $\nu = e^{-\lambda x^0} \mu(x^i)$. Then one find the following equations for λ and μ:

$$\left\{\begin{array}{ll}
(1) & \big(\partial \dot{x}_k.F^0\big)_s \mu^k + \big(\partial \dot{x}_k^p.F^0\big)_s \mu^k_p = 0 \\
(2) & \big(\partial \dot{x}_k.F^0_\alpha\big)_s \mu^k + \big(\partial \dot{x}_k^p.F^0_\alpha\big)_s \mu^k_p + \big(\partial \dot{x}_k^{pr}.F^0_\alpha\big)_s \mu^k_{pr} = \lambda \delta^0_\alpha G^i_{ik} \mu^k \\
(3) & \dfrac{1}{\rho}\big[\big(\partial \dot{x}_k.F^j\big)_s \mu^k + \big(\partial \dot{x}_k^p.F^j\big)_s \mu^k_p + \big(\partial \dot{x}_k^{ir}.F^j\big)_s \mu^k_{ir} + \big(\partial p^i.F^j\big)_s \mu^{(p)}_i\big] = \lambda \mu^j \\
(4) & \dfrac{1}{\rho C_p}\big[\big(\partial \dot{x}_k.F^4\big)_s \mu^k + \big(\partial \dot{x}_k^p.F^4\big)_s \mu^k_p + \big(\partial \theta^k.F^4\big)_s \mu^{(\theta)}_k + \big(\partial \theta^{ir}.F^4\big)_s \mu^{(\theta)}_{ir}\big] = \lambda \mu^{(\theta)}
\end{array}\right\}_{1 \leqslant j \leqslant 3}.$$

Thus the problem is reconduced to an eigenvalue problem (Eqs. (3)–(4)) conditioned by the constraints (1)–(2). The linear differential operator involved is not, in general, self-adjoint. So the eigenvalues have both real and imaginary parts. If the real parts are all positive, then the amplitude of every perturbation decays exponentially with time and the underlying stationary solution s is then said to be *linearly stable*. In general, to a stationary solution will correspond eigenvalues with negative and positive real parts. Then the smallest Reynolds number coresponding to steady-state solutions with eigenvalues having positive real part is called *critical linear Reynolds number* $R^{\mathrm{linear}}_{e,c}$. Of course solutions corresponding to eigenvalues with higher Reynolds numbers will be unstable. For non-stationary solutions we shall utilize the linearized equation $\widehat{(NS)_s}$ with a different approach. In order to see this, let us emphasize that $\widehat{(NS)_s}$

[6] Note that in order to apply Lemma 3.3 it is necessary that the symbol $\widehat{g}_{2+1}[s]$ be involutive. This is just the case as $\widehat{g}_2[s]$ is involutive and hence also $\widehat{g}_{2+r}[s]$, $\forall r \geqslant 0$, are involutive for definition.

can be considered a vector neighbourhood of $D^2s(M)$ into $\widehat{(NS)}$ and that $E \equiv s^*vTW$ is a vector neighbourhood of $s(M)$ into W. So a solution ν of $\widehat{(NS)_s}$ can be identified with a vertical shift of the section s of $\pi : W \to M$ into a *perturbated section* $s' \equiv s + \nu$. Then, we can say that s is *stable* if any of such admissible perturbations ν of s is such that the time-depending function $\mathfrak{p}(t) = \frac{1}{2}\int_{\underline{B}_t} \nu^2\eta / \int_{\underline{B}_t} \eta$ is a decreasing function. Note that the vector fiber bundle $\pi : E \to M$ has a natural Euclidean structure on the fibers as these are identified with $\mathbf{S} \times \mathbb{R}^2$. In fact one has the canonical isomorphism: $s^*vTW \cong M \times \mathbf{S} \times \mathbb{R}^2$, for any section $s : M \to W$. We call such a function the *square mean perturbation*. We say that a solution $V \subset (NS)$ is *t-unstable* if the perturbations arised at some instant t, on the 3-dimensional compact space-like submanifold $B_t \subset V$, produce a square mean perturbation $\mathfrak{p}(t)$ that increases in the time. Otherwise we say that V is *t-stable*. If V is t-stable at any $t \in T$, we say that it is *globally stable* (otherwise *globally unstable*). One has: $\mathfrak{p}(t) = \frac{1}{2\,\mathrm{vol}(\underline{B}_t)}[\int_{\underline{B}_t} (\nu^{(v)})^2\eta + \int_{\underline{B}_t} (\nu^{(p)})^2\eta + \int_{\underline{B}_t} (\nu^{(\theta)})^2\eta] \equiv \mathfrak{p}^{(v)} + \mathfrak{p}^{(p)} + \mathfrak{p}^{(\theta)}$. Let us, now, consider the rate $\dot{\mathfrak{p}}$ of variation of $\mathfrak{p}$ along the flow:[7]

$$(\Delta):\quad \dot{\mathfrak{p}}(t) = \frac{1}{2}\lim_{\lambda\to 0}\left(\frac{\int_{\phi_\lambda(\underline{B}_t)} \nu^2\eta}{\int_{\phi_\lambda(\underline{B}_t)} \eta} - \frac{\int_{\underline{B}_t} \nu^2\eta}{\int_{\underline{B}_t} \eta}\right)\Big/\lambda = \frac{1}{2}\frac{1}{\mathrm{vol}(\underline{B}_t)}\int_{\underline{B}_t} \frac{\phi_\lambda^*\nu^2 - \nu^2}{\lambda}\eta = \frac{1}{2}\frac{1}{\mathrm{vol}(\underline{B}_t)}\int_{\underline{B}_t} \frac{\delta\nu^2}{\delta t}\eta$$
$$\equiv \dot{\mathfrak{p}}^{(v)} + \dot{\mathfrak{p}}^{(p)} + \dot{\mathfrak{p}}^{(\theta)}.$$

Then a sufficient condition for the stability is that (♣): $\dot{\mathfrak{p}}(t) \leqslant -c\mathfrak{p}(t)$, where c is a positive constant. In fact, if we consider neglectable the perturbations in temperature and pressure, then for any admissible regular solution s of $\widehat{(NS)}$ the ratio $\dot{\mathfrak{p}}/\mathfrak{p}$ is given by the following formula: $\frac{\dot{\mathfrak{p}}(t)}{\mathfrak{p}(t)} = 2\frac{-[\int_{\underline{B}_t} (\nabla_{\nu^{(v)}} v).\nu^{(v)}\eta + \frac{\chi}{\rho}(d\underline{\nu}^{(v)}, d\underline{\nu}^{(v)})]}{\int_{\underline{B}_t} (\nu^{(v)})^2\eta}$. Then, if the solution $V \subset \widehat{(NS)}$ is such that there exists a constant $c > 0$, such that $\int_{\underline{B}_t} (\nu^{(v)})^2\eta \leqslant \int_{\underline{B}_t} [(\nabla_{\nu^{(v)}} v) \cdot \nu^{(v)} - \frac{\chi}{\rho}(\overline{\Delta\underline{\nu}^{(v)}}) \cdot \nu^{(v)}]\eta$, for any admissible perturbation $\nu^{(v)}$ of the velocity field of the space–time flow, then the solution V is stable, assumed that the perturbations of temperature and pressure are neglectable. Let $V \subset \widehat{(NS)}$ be a solution such that the perturbations of the temperature and pressure are neglectable. Then V is stable if the smallest eigenvalue $\bar{\lambda}_1 = \bar{\lambda}_1(t)$ of the self-adjoint differential operator $P[s](\nu) = -\frac{\chi}{\rho}\Delta\underline{\nu}^{(v)} + \langle\dot{e}, \nu^{(v)}\rangle + dp$, for divergence free vector fields $\nu^{(v)}$, canonically associated to the solution s, is positive for any $t \in T$ and lower bounded: $\bar{\lambda}_1(t) \geqslant \lambda_1 \geqslant 0$. Furthermore, let us assume also that the domain $\underline{B}_t$ has the negative Laplacian $-\Delta$ for differential 1-forms, along with boundary conditions, strictly self-adjoint, with a discrete spectrum and smallest eigenvalue $\lambda_1(t) > 0$, that is lower bounded: $\lambda_1(t) \geqslant \lambda_1 \geqslant 0$, then a sufficient condition for the stability of the solution $V \subset \widehat{(NS)}$ is that the Reynolds number $R_e^{\sup}$, considered as the upper bound of the characteristic Reynolds number $R_e(t)$, satisfies the following condition: (◇◇): $(R_e^{\sup})^2 \leqslant \lambda_1$. Note that if condition (◇◇) is satisfied, then as the characteristic Reynolds number $R_e < R_e^{\sup}$, also $(R_e)^2 \leqslant \lambda_1$ holds. We call *non-linear critical Reynolds number* the adimensional number $R_{e,c}^{\text{non-linear}}$ such that $(R_{e,c}^{\text{non-linear}})^2 = \lambda_1$. Note that we denote also $R_e^{\sup}(t) = U_{\sup}(t)L\rho/\chi$, where $U_{\sup}(t) = \sup_{p\in\underline{B}_t} |v(p)|$. Taking into account the *Poincaré's inequality* [5] [Let X be a 3-dimensional compact orientable Riemannian manifold. Let $\underline{\nu} \in \Omega^1(X)$ be a differential 1-form such that $\underline{\nu}^2$ and $(d\underline{\nu})^2$ are integrable on X. Then, one has: $\frac{-\int_X (\overline{\Delta\underline{\nu}})\cdot\nu\eta}{\int_X \underline{\nu}^2\eta} = \frac{(d\underline{\nu}, d\underline{\nu})}{\int_X \underline{\nu}^2\eta} = \frac{\int_X (d\underline{\nu})^2\eta}{\int_X \underline{\nu}^2\eta} \geqslant \lambda_1$, where $\lambda_1 > 0$ is the smallest eigenvalue of $-\Delta$.] we can conclude that a regular solution $V \subset \widehat{(NS)} \subset JD^2(W)$ is stable if the upper bound $R_e^{\sup}$ of the characteristic Reynolds number R_e satisfies the inequality $(R_e)^2 \leqslant \lambda_1$. As a by-product of above results we have that in a same solution of (NS) may be present 3-dimensional space-like domains, B_t where the Reynolds number $R_e^{\sup}(t)$, upper bound of the Reynolds characteristic number $R_e(t)$, at the section B_t, exceds the critical value $R_{e,c}^{\text{non-linear}}$ other the which one has unstability, and other ones where, instead, the $R_e^{\sup}(t)$ is lower than the critical value. (In such cases $R_e^{\sup} \geqslant R_e^{\text{non-linear}}$ and so the solution is globally unstable.) Note that if ν is a solution of $\widehat{(NS)_s}$, then $\nu' \equiv \phi^*\nu = vT(\phi) \circ \nu \circ \bar{\phi}^{-1}$ is a solution of $\widehat{(NS)_{s'}}$, $s' = \phi^*s$, $\forall\phi \in G =$ pseudogroup of symmetries of (NS). Then assuming that thermal and pressure perturbations are neglectable for s, and that s is stable, then also s' is stable. In fact, $\forall\phi \in G$ one has: $\mathfrak{p}'(t) = \frac{\int_{\underline{B}_t} (\nu'^{(v)})^2\eta}{\mathrm{vol}(\underline{B}_t)} = \frac{\int_{\underline{B}_t} \langle\phi^*\nu^{(v)}, \phi^*\nu^{(v)}\rangle\eta}{\mathrm{vol}(\underline{B}_t)} = \frac{\int_{\underline{B}_t} \phi^*g(\nu^{(v)}, \nu^{(v)})\eta}{\mathrm{vol}(\underline{B}_t)} = \frac{\int_{\underline{B}_t} (\nu^{(v)})^2\eta}{\mathrm{vol}(\underline{B}_t)} = \mathfrak{p}(t)$, $\forall t \in T$. So the symmetry pseudogroup G of $\widehat{(NS)}$ allows us to transform stable solutions into stable solutions. Furthermore,

[7] Note that the above formula for $\mathfrak{p}$, works well also if the flow ϕ_λ is not a diffeomorphism for all λ, i.e., if there are singular points in the flow.

$\dot{\mathfrak{f}}$ is, in general, a function of the time, $\dot{\mathfrak{f}} = \dot{\mathfrak{f}}(t)$. So the Reynolds number is a characteristic number of the solution V iff $\dot{\mathfrak{f}} = 0$. In such a case if $R_e \leqslant R_{e,c}^{\text{non-linear}}$, at the initial instant, then the solution is stable assumed that the thermal and pressure perturbations are neglectable.[8] □

Remark 3.4. From above results it follows that it is clear that smoothness for a solution $V \subset (NS)$, does not necessarily imply absence of singularities in the corresponding flow ϕ_λ generated by the characteristic vector field $\zeta = \partial\phi$ associated to the solution V, and by means of the which the initial Cauchy data $B_{t_0} \subset V$ is propagated to generate all the solution $V = \bigcup_\lambda \phi_\lambda(B_{t_0})$. In other words smoothness of the solution is not synonymous of absence of singularity in the flow (*characteristic singularity*). So we should modify the question put in the introduction of the book by Doering & Gibbon: "*It is possible that the equations produce solutions which exhibit finite-time singularities.*" [5], with the following more mathematically precise one: "*It is possible that* $\widehat{(NS)}$ *produces solutions which exhibit finite time instability*?"

In the following Theorems 3.5 and 3.7 we give mathematically rigorous answers to this question. (See also Ref. [15] for a different characterization of instability related to Ulam instability of functional equations.)

Theorem 3.5 *(Full dynamic characterization of global smooth solutions of* $\widehat{(NS)}$*).*

(1) *If* $V \subset (NS) \subset J_4^2(W)$ *is a smooth (global) solution without characteristic singularities, and such that the vorticity is harmonic,* $\Delta\underline{\omega} = 0$ *(or equivalently* $\operatorname{rot}\omega = 0$*, in particular if* $\underline{\omega} = 0$*), then the vorticity is conserved along the flow. The existence of non-trivial harmonic vorticity at any instant is related to the topology of the space-like 3-dimensional manifold* B_t *and its boundary* ∂B_t.[9] *If the Reynolds number* $R_e^{\sup}$ *is lower than the critical value* $R_{e,c}^{\text{non-linear}}$*, then* V *is a globally stable solution (assumed that the thermal and pressure perturbations are neglectable). Furthermore, if* $\dot{\mathfrak{f}}(V) = 0$*, the characteristic Reynolds number* R_e *is a characteristic number of the solution, and the instability cannot arise if* $R_e < R_{e,c}^{\text{non-linear}}$*; otherwise the solution becomes finite-time instable.*
(2) *If* $V \subset (NS)$ *is a smooth (global) solution having some characteristic singularities, then the vorticity cannot be conserved, even if it is harmonic. If the initial Reynolds number* $R_e^{\sup}$ *is lower than the critical value* $R_{e,c}^{\text{non-linear}}$*, then* V *is a globally stable solution (assumed that the thermal and pressure perturbations are neglectable). Furthermore, if* $\dot{\mathfrak{f}}(V) = 0$*, the characteristic Reynolds number* R_e *is a characteristic number of the solution, and the instability cannot arise if* $R_e < R_{e,c}^{\text{non-linear}}$*; otherwise the solution becomes finite-time instable.*
(3) *If* $D = 2$ *and the vorticity is harmonic (or equivalently* $\operatorname{rot}\omega = 0$*, in particular if* $\underline{\omega} = 0$*), then it is conserved for any smooth (global) solution of the Navier–Stokes equation. Really, in this case we have* $\omega = h\partial x_3$*,* $h \in \mathbb{R}$*. (In other words, the fluid has the skew symmetric part,* ω*, of the full infinitesimal strain rate,* $\dot{\mathfrak{e}} = \dot{e} + \omega$*, that is a simply rigid rotation.)*

Proof. (1) The *vorticity equation* for the Navier–Stokes equation can be written in the form $(\diamond)$ $\rho\frac{\delta\omega}{\delta t} = \rho\nabla_\omega u + \chi\overline{\Delta\underline{\omega}}$ or equivalently $(\underline{\diamond})$ $\rho L_v\omega = \chi\overline{\Delta\underline{\omega}}$.[10] Then if $\Delta\underline{\omega} = 0$ one has $L_v\omega = 0$. As the flow is without characteristic singularities, above equation is equivalent to say $\underline{\phi}_\lambda^*\omega = \omega$, $\forall\lambda$, where $\partial\underline{\phi} = v$. In fact, in such cases $\underline{\phi}_\lambda$ is a diffeomorphism between the related 3-dimensional integral space-like manifolds $\overline{B}_t \subset V$. To conclude the proof of the point (1) of the theorem it is enough to consider results obtained in the proof of Theorem 3.2.

(2) In fact, even if one has $\Delta\underline{\omega} = 0$, hence $L_v\omega = 0$, as the flow is not a diffeomorphism between the related 3-dimensional integral space-like manifolds $B_t \subset V$, from the equation $L_v\omega = 0$ we cannot argue that it is also

[8] This is, for example, the case of a laminar flow in a circular pipe [11,13].

[9] Note that $\ker(\Delta) \cong H^2(B_t;\mathbb{R}) \cong \mathbb{R}^{\sharp\operatorname{comp}\partial B_t - \sharp\operatorname{comp} B_t}$, and $\dim H^2(B_t;\mathbb{R}) = \sharp\operatorname{comp}\partial B_t - \sharp\operatorname{comp} B_t$. Then, the existence of non-trivial (i.e., $\underline{\omega} \neq 0$), harmonic vorticity at the instant t, is related to the topology of the space-like 3-dimensional manifold B_t and its boundary ∂B_t. For example, if $B_t \simeq D^3$ (3-dimensional disk), then $\partial B_t \simeq S^2$ (2-dimensional space-like sphere). Since $H^2(B_t;\mathbb{R}) = 0$, it follows that all harmonic vorticities are trivial. So, if the boundary conditions impose that this topology of B_t should be conserved at any t, and there is not vorticity at some instant, we will have an irrotational flow. Furthermore, if B_t is the region between two concentric round sphere, then $H^2(B_t;\mathbb{R}) = \mathbb{R}$, hence $\dim\ker(\Delta) = 1$. In such a case we can have some non-trivial harmonic vortices. Therefore, if the fixed boundary conditions impose that such a topology is conserved at any instant, we will have a conserved non-zero vorticity iff it is harmonic. (See, e.g., Ref. [4].)

[10] The term $\nabla_\omega u$ is called the *vortex stretching*. Here $\frac{\delta}{\delta t}$ is the symbol of covariant derivative along the space–time flow. Furthermore, L_v is the symbol of Lie derivative with respect to the space–time velocity v of the fluid.

$\underline{\phi}^*_\lambda \omega = \omega$, $\forall\lambda$. Emphasize, also, that even if equation ($\diamond$) can be rewritten in covariant form: ($\bar{\diamond}$) $\rho \frac{\delta \underline{\omega}}{\delta t} = \rho \nabla_\omega \underline{u} + \chi \Delta \underline{\omega}$, equation ($\bar{\diamond}$) is not equivalent to the following equation: $\rho L_v \underline{\omega} = \chi \nabla \underline{\omega}$. Note that in covariant form, the condition $L_v \underline{\omega} = 0$ is really equivalent to $\underline{\phi}^*_\lambda \underline{\omega} = \underline{\omega}$, $\forall\lambda$, even if $\underline{\phi}_\lambda$ is not a diffeomorphism for any λ. Therefore, when on V there are singular points, in the flow of the characteristic vector field, the presence of the vortex stretching term $\nabla_\omega \underline{u}$ does not allow that the vorticity should be conserved even if it is harmonic. Now, by using results in the proof of Theorem 3.2 we conclude the proof of point (2).

(3) In fact, when the space-like domain B, where the fluid is defined at some instant, is 2-dimensional, orientable, and it is constrained to stay in a plane σ, then the vorticity can be written in the form $\underline{\omega}|_{\underline{B}_t} = h\eta_\sigma$, where h is, at any instant, a scalar function on σ and η_σ is the volume 2-form on σ. Then, the vorticity equation can be written as follows: $\rho \eta_\sigma \frac{\delta h}{\delta t} = \chi \Delta(h\eta_\sigma)$. Therefore the condition of conservation for the vorticity is equivalent to require the harmonicity of $\underline{\omega}$. This is equivalent to ask that $d\underline{\omega} = 0$ and $\delta \underline{\omega} = 0$. Now, we can consider a system of coordinates, adapted to the inertial frame, such that the spatial coordinates $\{x^k\}_{1 \leqslant k \leqslant 3}$, are adapted also to the constraint for the fluid, i.e., such that $\partial x_3 \perp \sigma$. Then the vertical metric g can be written: $g_{ab} = \gamma_{ab} = \gamma_{ab}(x^1, x^2)$, $a, b = 1, 2$, $g_{ij} = \delta_{ij}$, if at least one of the indexes i and j is different from a or b. Here (γ_{ab}), is the metric on σ. Then $\eta_\sigma = \sqrt{|\gamma|}\, dx^1 \wedge dx^2$, $|\gamma| = \det(\gamma_{ab})$, is the volume form on σ and the vorticity can be represented by $\underline{\omega} = h\eta_\sigma$. The first condition for the harmonicity of $\underline{\omega}$ is $0 = d\underline{\omega} = dh \wedge \eta_\sigma$, hence $dh = 0$. The second condition for the harmonicity of $\underline{\omega}$ gives: $0 = \delta\underline{\omega} = \delta(h\eta_\sigma) = \star^{-1} d \star (h\eta_\sigma) = \star^{-1} d(h.dx^3) = \star^{-1}(dh \wedge dx^3)$, hence $dh = 0$. Here $\star$ is the $\star$-Hoodge operator [10]. Therefore, $\underline{\omega}$ is harmonic iff $\underline{\omega} = h\eta_\sigma$, with $h = h(t)$. But for such a function the vorticity equation requires $(\partial t.h) = 0$. This is equivalent to say that $\omega = h\partial x_3$, $h \in \mathbb{R}$. □

Corollary 3.6. *By simply fixing the boundary conditions we cannot, in general, determine the conservation of the vorticity.*

Proof. In fact, from Theorem 3.2 it follows that for any boundary condition we have not, in general, an unique solution. Hence, from Theorem 3.5 it follows that we can have solutions where the vorticity is conserved and other ones where, instead it is not conserved.[11] □

Theorem 3.7 *(Existence of smooth global solutions without characteristic singularities). Let us assume that at some instant t_0 the fluid is contained into a compact 3-dimensional smooth manifold B_t that is p-connected, $0 \leqslant p \leqslant 2$, and has the homotopy type of the 3-dimensional disk D^3,[12] and with smooth boundary, that evolves with a fixed law in such a way to describe a 3-dimensional smooth time-like manifold, that is q-connected, $0 \leqslant q \leqslant 1$, and such that the sectional topology is conserved (i.e., it remains D^3), and that is at the rest with respect to the inertial frame. Let us assume also the non-slip condition of the fluid at the boundary. Then there exist global smooth solutions of (NS) that have the corresponding characteristic flows without singularities. In general, such solutions V are not unique and are not necessarily stable. However, if the Reynolds number $R_e^{\mathrm{sup}} \leqslant R_{e,c}^{\text{non-linear}}$ and the thermal and pressure perturbations are neglectable, then V is globally stable. Furthermore, if $\dot{\mathfrak{f}}(V) = 0$, the characteristic Reynolds number R_e is a characteristic number of the solution V too, and the instability cannot arise if $R_e < R_{e,c}^{\text{non-linear}}$; otherwise the solution becomes finite-time instable.*

Proof. Note that, from the topological point of view, such a solution is D^4 and its boundary is S^3. It is well known that on S^3 there are nowhere zero vector fields. In fact, from the Hopf's theorem it follows that a smooth connected orientable closed n-dimensional manifold X has a non-zero tangent vector field iff $\chi(X) = 0$, where $\chi(X)$ is the Euler characteristic number of X. On the other hand as $\chi(X) = \sum_i (-1)^i \dim H_i(X)$, it follows that $\chi(S^3) = \dim H_0(S^3) - \dim H_3(S^3) = 1 - 1 = 0$. Furthermore, it is well known (see e.g. [4,7]) that if $X \in [\emptyset] \in \Omega_n$, i.e., $X = \partial V$, then $\chi(X) = 2m$ and if $\dim X = 2s + 1$ one has $\chi(X) = 0$. This is a necessary condition that V has a non-zero

[11] This solves the "non-analyticity paradox," i.e., in order to require vorticity, a region of fluid initially at rest (or in irrotational motion) must have a vorticity which is a non-analytic function of time [3]. In fact, in solutions with singularities (solutions considered in Theorem 3.5(2)) the vorticity can start from zero and become different from zero, and again vanish and so on. Such solutions cannot, in fact, be realized as analytic sections of $\pi : W \to M$.

[12] More precisely B_t has the following homotopy and homology groups, respectively: $\pi_0(B_t) \cong \pi_p(B_t) \cong H_p(B_t; \mathbb{Z}) = 0$, $1 \leqslant p \leqslant 3$; $H_0(B_t; \mathbb{Z}) \cong \mathbb{Z}$.

vector field. In fact if V is a compact smooth manifold with $\chi(V) \neq 0$, then any tangent vector field ζ on V has a zero. (Index$(\zeta) = \chi(V)$.) On the other hand any compact connected manifold with boundary has a non-vanishing vector field. (See, e.g., Ref. [7].) Therefore, on such a solution V the characteristic vector field can be without zeros. Furthermore, as in such case one has that the space H^1 of harmonic 1-forms is zero ($H^1 = H_1(B_t; \mathbb{R}) \cong H_1(S^2; \mathbb{R}) = 0$), it follows that the unique harmonic vorticity is the zero vorticity. Then, from Theorem 3.6 we conclude that there exists a laminar (irrotational) (smooth) global solution. Furthermore, taking into account that the non-slip hypothesis assures that all the integral characteristic numbers are zero on any compact closed boundary ∂V at any instant t,[13] we can conclude that exist smooth solutions. However, in general, such solutions are not unique. In fact, if $s : X \subset M \to W$ is such a solution, we can consider the shifted smooth solution $s' = s + \nu$ such that $\nu : X \to s^*vTW$, $\nu|_{\partial X} = 0$ and $D^2\nu : X \to \widehat{(NS)_s} \subset \widehat{(NS)}$. So $D^2 s' : X \to (NS)$. Note that the existence of such solutions is assured as just proved in the proof of Theorem 3.2. Note also, that the existence of regular global smooth solutions of $(NS) \subset J^2_4(W)$, i.e., solutions diffeomorphic to 4-dimensional smooth submanifolds of W, but not necessarily diffeomorphic to their projections on M, guarantees the non-uniqueness of such solutions. Finally, by using the full Theorem 3.2 we can conclude the proof. □

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[13] See Theorem 2.6 and Example 2.7.