Existence and Smoothness of Navier-Stokes Equations

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Abstract

In this paper we propose new method for proving of existence of global solutions for 3D Navier-Stokes equations. This complies an application to the Clay Institute Millennium Prize Navier Stokes Problem. The proposed method can be applied for investigation of global solutions for other classes of PDEs.

Key words: Navier-Stokes equations, global existence.

AMS subject classification: Primary 35Q30, 76D05, 46E35; Secondary 35B65, 35K55.

1 Introduction

In this article we investigate the following Navier-Stokes Equations (1.1)

$$\begin{cases} u_t + uu_x + vu_y + wu_z + \dfrac{1}{\rho} p_x - \vartheta u_{xx} - \vartheta u_{yy} - \vartheta u_{zz} = 0 \\ v_t + uv_x + vv_y + wv_z + \dfrac{1}{\rho} p_y - \vartheta v_{xx} - \vartheta v_{yy} - \vartheta v_{zz} = 0 \\ w_t + uw + vw_y + ww_z + \dfrac{1}{\rho} p_z - \vartheta w_{xx} - \vartheta w_{yy} - \vartheta w_{zz} = 0 \\ u_x + v_y + w_z = 0 \quad \text{in } (0,\infty) \times R^3 \\ u(0,x,y,z) = u_0(x,y,z), \quad v(0,x,y,z) = v_0(x,y,z) \\ w(0,x,y,z) = w_0(x,y,z) \ in \ R^3, \end{cases}$$

where $p,u,v,w : [0,\infty) \times R^3 \to R$ are unknown, u_0, v_0 $w_0 \in C^\infty(R^3)$ are given functions.

This is a system of partial differential equations that governs the flow of a viscous incompressible fluid. ρ is the density, u the velocity vector, P the pressure. The first three equations of (1.1) are Cauchy's momentum equations where the first term is the accelerating time varying term, the second and third are the convective and the hydrostatic terms respectively. The physical example of the convective term can be described as a river that is converging, the case where the term is increasing and the river diverging the case where the term is decreasing. The hydrostatic term describe flow from high pressure to low pressure. The forth term is the viscosity term with the coefficient ν the kinematical viscosity. This term is said to describe the ability of the fluid to induce motion of neighboring particles. On the right hand side we have the external forces density term. This term can include: gravity, magneto-hydrodynamic force, and so on. The fourth equation of (1.1) is the nullification of the divergence due to incompressibility condition. Turbulent fluid motions are believed to be well modeled by the Navier-Stokes equations. But, due to the complexity of the equations most of our understanding relies on laboratory experiments. This is the main reason why it is necessary to know basic features of the equations like existence and smoothness of the equations' solutions. In the case of the 3D version of the NS equations the existence problem is an unsolved issue([1]).

We recall that global existence of weak solutions of the Naiver-Stokes equations is known to hold in every space dimension. Uniqueness of weak solutions and global existence of strong solutions is known in dimension two [4]. In dimension three, global existence of strong solutions of the NavierStokes equations in thin three-dimensional domains began with the papers [5] and [6], where is used the methods in [2] and [3].

In this paper we propose new method for investigation of equations (1.1). The proposed method gives existence of classical solutions for the problem (1.1).

Without loss of generality we assume that $\rho = \nu = 1$. For a set A such that $A \subset R$ or $A \subset R^2$ or $\subset R^3$, with μ(A) we will denote it measure.

Let $R^3 = U^\infty{}_{j=1} \ D_j$, where D_j be bounded subsets of R^3 satisfying the following conditions

1. $D_i \cap D_j = \emptyset$ for $i \neq j$,, $i,j \in \{1,2,...\}$,
2. D_j and D_{j+1}, $j \in \{1,2,...\}$, are adjoining,

3. if $(x_1, y_1, z_1) \in D_j$ *for some* $j \in \{1,2,\dots\}$, *is fixed and*

$D_{jx_1} = \{(y,z) \in R^2 : (x_1,y,z) \in D_j \}$,
$D_{jy_1} = \{(x,z) \in R^2 : (x, y_1,z) \in D_j \}$,
$D_{jz_1} = \{(x,y) \in R^2 : (x, y, z_1) \in D_j \}$,
$D_{jx_1y_1} = \{ z \in R : (x_1, y_1, z) \in D_j \}$,
$D_{jx_1z_1} = \{ y \in R : (x_1, y\text{ -}, z_{1}1) \in D_j \}$,
$D_{jy_1z_1} = \{ x \in R : (x, y_1, z_1) \in D_j \}$,

We have

$\mu(D_{jx_1}) \le \mu(D_j)$,
$\mu(D_{jy_1}) \le \mu(D_j)$,
$\mu(D_{jz_1}) \le \mu(D_j)$,
$\mu(D_{jx_1y_1}) \le \mu(D_j)$,
$\mu(D_{jx_1z_1}) \le \mu(D_j)$,
$\mu(D_{jy_1z_1}) \le \mu(D_j)$, $j \in \{1,2,...\}$.

For every $j \in \{1,2,...\}$ we denote with D_{jj} a compact subset of D_j such that $D_{jj} \neq D_j$.
for a set $B \subset R^3$ *and a function* $f\colon R^3 \to R$ *with* $f|_B$ *we denote the restriction of* f *to* B.

Our main result is as follows

Theorem 1.1. *Let* $u_0, v_0, w_0 \in C^\infty(R^3)$ *be such that*
$u_0|_{D_j}, v_0|_{D_j}, w_0|_{D_j} \in C^\infty(D_j)$,
$\text{supp}\, u_0|_{D_j}, \text{supp}\, v_0|_{D_j}, \text{supp}\, w_0|_{D_j} \subset D_{jj} \in \{1,2,...\}$,

and

$$\left|\partial_x^{\alpha 1}\partial_y^{\alpha 2}\partial_z^{\alpha 3} u_0\,(x,y,z)\ \right| \le C\alpha_1\alpha_2\alpha_3 K\left(1+\sqrt{x^2+y^2+z^2}\right)^{-k},$$
$$\left|\partial_x^{\alpha 1}\partial_y^{\alpha 2}\partial_z^{\alpha 3} v_0(x,y,z)\ \right| \le C\alpha_1\alpha_2\alpha_3 K\left(1+\sqrt{x^2+y^2+z^2}\right)^{-k},$$
$$\left|\partial_x^{\alpha 1}\partial_y^{\alpha 2}\partial_z^{\alpha 3} w_0(x,y,z)\right| \le C\alpha_1\alpha_2\alpha_3 K\left(1+\sqrt{x^2+y^2+z^2}\right)^{-k}, (1.2)$$

on R^3, *for any* $\alpha_1, \alpha_2, \alpha_3 \in N_0$ *and positive constant K.*

Then the problem (1.1) has a solution $(u,v,w,p) \in (C^\infty([0,\infty) \times R^3))^4$ *such that*

$$\int_{R^3}|u(t,x,y,z)|^2\,dx\,dy\,dz \le C_1 \quad \int_{R^3}|v(t,x,y,z)|^2\,dx\,dy\,dz \le C_1$$

$$\int_{R^3}|w(t,x,y,z)|^2\,dx\,dy\,dz \le C_1 \quad \int_{R^3}|p(t,x,y,z)|^2\,dx\,dy\,dz \le C_1$$

for some positive constant C_1.

Remark 1.2 $If\ u_0 \not\equiv 0, v_0 \not\equiv 0, w_0 \not\equiv 0$, then we obtain a nontrivial solution of the system (1.1).

2 Preliminaries

Definition 2.1 $let\ (X,d) be\ a\ metric\ space\ and\ M\ be\ a\ subset\ of\ X.\ The$

$mapping\ T: M \mapsto X\ is\ said\ to\ be\ expansive\ if\ there\ axists\ a\ constant\ h > 1$

$such\ that$

$$d(Tx, ty) > hd(x, y)$$

$for\ any\ x, y \in$ M.

Theorem 2.2 ([7], Theorem 2.4). Let X be a nonempty closed convex subset of a Banach space E. Suppose that T and S map X into E such that

1. S is continuous and S(X) resides in a compact subset of E.
2. T:X ↦E is expansive.
3. $S(X) \subset (I-T)(E) and[x = Tx + Sy,$
 $y \in$ X] $\rightarrow x \in$ X (or S(X) ⊂ (I − T)(X)).

then there exist a point $x^* \in$ X such that

$$Sx^* + Tx^* = x^*$$

Theorem 2.3. Let X be a nonempty closed convex subset of a Banach space E and Y is a nonempty compact subset of E such that X⊂ Y , Y ≠ X.
Suppose that T and S map X into E such that

1. S is continuous and S(X) resides in Y.

2. T : X ↦E is linear, continuous and expansive, and T : X ↦Y is onto, and {x - z : x∈ X; z ∈ S(X)} ⊂Y .
Then there exists an x_ 2 X such that

$$Sx^* + Tx^* = x^*$$

Proof. Since Y is compact and S(X) resides in Y, we have that the first condition of Theorem 2.2 holds. Because T: X ↦E is expansive, we have that the second condition of Theorem 2.2 holds. Note that T^{-1} : Y $\mapsto$E exists, it is linear and contractive with a constant $l \in (0,1)$. Let z ∈ S(X) be arbitrarily chosen and fixed. Set

$$A = \{y - z : y \in Y\}.$$

Take $y_0 \in$ Y arbitrarily. Define the sequence $\{y_n\}_{n \in N}$ as follows.

$y_{n+1} = T^{-1}y_n - z, \; n \in \mathbb{N} \cup \{0\}.$

Then

$$||y_2 - y_1|| = ||T^{-1}y_1 - T^{-1} y_0||$$
$$\leq l \, ||y_1 - y_0||,$$
$$||y_3 - y_2|| = ||T^{-1}y_2 - T^{-1} y_1||$$
$$\leq l \, ||y_2 - y_1||,$$
$$\leq l^2 \, ||y_1 - y_0||.$$

Using the principle of the mathematical induction, we get

$$||y_{n+1} - y_n||, \leq l^n \, ||y_1 - y_0||, \; n \in \mathbb{N}$$

Now, from m > n, m, n $\in \mathbb{N}$, we find

$$||y_m - y_n|| \leq ||y_m - y_{m-1}|| + \ldots + ||y_{n+1} - y_n||$$
$$\leq (l^{m-1} + \ldots + l^n)||y_1 - y_0||$$
$$\leq l^n \sum_{j=0}^{\infty} l^j \, ||y_1 - y_0||$$
$$= \frac{l^n}{1-l} \, ||y_1 - y_0||.$$

Therefore $\{y_n\}_{n \in N}$ is a Cauchy sequence of elements of Y⊂ E. Since E is a Banach space, it follows that the sequence $\{y_n\}_{n \in N}$ is convergent to an element $y^* \in$ E. Because $\{y_n\}_{n \in N} \subset$ Y and Y ⊂ E is compact, we have that $y^* \in$ EY. Thus

$$y^* = T^{-1}y^* - z$$

Or

$$z^* = T\, z^* + z, \; z^* = T^{-1}\, y^* \in X$$

Because z ∈ S(X) was arbitrarily chosen, we conclude that S(X) ⊂ (I - T) (X), i.e., the third condition of Theorem 2.2 holds. Hence Theorem 2.2, it follows that there exists an $x^* \in$ X such that

$$Tx^* + Sx^* = x^*.$$

This completes the proof.

3 Proof of the Main Result

From the fourth equation of the system (1.1), we get

$$u(u_x + v_y + w_z) = 0, \quad v(u_x + v_y + w_z) = 0, \quad w(u_x + v_y + w_z) = 0.$$

Then the system (1.1) we can rewrite in the form

$$\begin{cases} u_t + uu_x + vu_y + wu_z + u(u_x + v_y + w_z) + p_z - u_{xx} - u_{yy} - u_{zz} = 0 \\ v_t + uv_x + vv_y + wv_z + v(u_x + v_y + w_z) + p_z - v_{xx} - v_{yy} - v_{zz} = 0 \\ w_t + uw_x + vw_y + ww_z + w(u_x + v_y + w_z) + p_z - w_{xx} - w_{yy} - w_{zz} = 0 \\ u_x + v_y + w_z = 0 \quad \text{in } (0,\infty) \times R^3 \\ u(0,x,y,z) = u_0(x,y,z), \quad v(0,x,y,z) = v_0(x,y,z) \\ w(0,x,y,z) = w_0(x,y,z) \; in \; R^3, \end{cases}$$

whereupon

$$\begin{cases} u_t + (u^2)_x + (uv)_y + (uw)_z + p_x - u_{xx} - u_{yy} - u_{zz} = 0 \\ v_t + (uv)_x + (v^2)_y + (vw)_z + p_z - v_{xx} - v_{yy} - v_{zz} = 0 \\ w_t + (uw)_x + (vw)_y + (w^2)_z + p_z - w_{xx} - w_{yy} - w_{zz} = 0 \\ u_x + v_y + w_z = 0 \quad \text{in } (0,\infty) \times R^3 \\ u(0,x,y,z) = u_0(x,y,z), \quad v(0,x,y,z) = v_0(x,y,z) \\ w(0,x,y,z) = w_0(x,y,z) \; in \; R^3. \end{cases} \tag{3.1}$$

Remark 3.1. We note that using the fourth equation of (3.1) we can obtain the system (1.1).

Step 1. Let $j \in \{1,2,\dots\}$ be arbitrarily chosen. Firstly, we consider the Problem

$$\begin{cases} u_t + (u^2)_x + (uv)_y + (uw)_z + p_x - u_{xx} - u_{yy} - u_{zz} = 0 \\ v_t + (uv)_x + (v^2)_y + (vw)_z + p_y - v_{xx} - v_{yy} - v_{zz} = 0 \\ w_t + (uw)_x + (vw)_y + (w^2)_z + p_z - w_{xx} - w_{yy} - w_{zz} = 0 \\ u_x + v_y + w_z = 0 \quad \text{in } (0,1] \times D_j, \\ u(0,x,y,z) = u_0(0,x,y,z), \quad v(0,x,y,z) = v_0(0,x,y,z), \\ w(0,x,y,z) = w_0(0,x,y,z) \quad in \quad D_j. \end{cases} \tag{3.2}$$

We will prove that the problem (3.2) has a solution (u,v,w,p) such that u,v,w,p ∈ C^1 ([0,1], ${C_0}^2$ (D_j)).

Let $(x_0, y_0, z_0) \in D_j$ be arbitrarily chosen.

For *u,v,w,p* ∈ C^1 ([0,1], ${C_0}^2(D_j)$), we define

$$I_1^{1j}(u,v,w,p) = \int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1}(u(t,\alpha,\beta,\gamma) - u_0(\alpha,\beta,\gamma))\, d\gamma dz_1 d\beta dy_1 d\alpha dx_1$$

$$+\int_0^t\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1}(\, u^2(s,\alpha,\beta,\gamma))\, d\gamma dz_1 d\beta dy_1 d\alpha ds$$

$$+\int_0^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{z_0}^{z}\int_{z_0}^{z_1}u(s,\alpha,\beta,\gamma)v\,(s,\alpha,\beta,\gamma))\, d\gamma dz_1 d\beta d\alpha dx_1 ds$$

$$+\int_0^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}u(s,\alpha,\beta,\gamma)w\,(s,\alpha,\beta,\gamma))\, d\gamma d\beta dy_1 d\alpha dx_1 ds$$

$$+\int_0^t\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1}p(s,\alpha,\beta,\gamma)\, d\gamma dz_1 d\beta dy_1 d\alpha ds$$

$$-\int_0^t\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1}u(s,x,\beta,\gamma)\, d\gamma dz_1 d\beta dy_1 ds$$

$$-\int_0^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{z_0}^{z}\int_{z_0}^{z_1}u(s,\alpha,y,\gamma)\, d\gamma dz_1 d\alpha dx_1 ds$$

$$-\int_0^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{z_0}^{y}\int_{z_0}^{y_1}u(s,\alpha,\beta,z)\, d\beta dy_1\, d\alpha dx_1 ds$$

$$I_2^{1j}(u,v,w,p) = \int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1}(v(t,\alpha,\beta,\gamma) - v_0(\alpha,\beta,\gamma))\, d\gamma dz_1 d\beta dy_1 d\alpha dx_1$$

$$+\int_0^t\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1}u\,(s,\alpha,\beta,\gamma)v\,(s,\alpha,\beta,\gamma)\, d\gamma dz_1 d\beta dy_1 d\alpha ds$$

$$+\int_0^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{z_0}^{z}\int_{z_0}^{z_1}v^2(s,\alpha,\beta,\gamma)\, d\gamma dz_1 d\beta dy_1 d\alpha dx_1 ds$$

$$+\int_0^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}v(s,\alpha,\beta,\gamma)w\,(s,\alpha,\beta,\gamma))\, d\gamma d\beta dy_1 d\alpha dx_1 ds$$

$$+\int_0^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{z_0}^{z}\int_{z_0}^{z_1}p(s,\alpha,\beta,\gamma)\, d\gamma dz_1 d\beta d\alpha dx_1 ds$$

$$-\int_0^t\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} v(s,x,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1 ds$$

$$-\int_0^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{z_0}^{z}\int_{z_0}^{z_1} v(s,\alpha,y,\gamma)\,d\gamma dz_1 d\alpha dx_1 ds$$

$$-\int_0^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1} v(s,\alpha,\beta,z)\,d\beta dy_1\, d\alpha dx_1 ds,$$

$$I_3^{1j}\,(u,v,w,p) = \int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1}(w(t,\alpha,\beta,\gamma) - w_0(\alpha,\beta,\gamma))\,d\gamma dz_1 d\beta dy_1 d\gamma dx_1$$

$$+\int_0^t\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u\,(s,\alpha,\beta,\gamma)w\,(s,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1 d\alpha ds$$

$$+\int_0^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{z_0}^{z}\int_{z_0}^{z_1} v(s,\alpha,\beta,\gamma)w\,(s,\alpha,\beta,\gamma))\,d\gamma dz_1 d\beta d\alpha dx_1 ds$$

$$+\int_0^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z} w^2(s,\alpha,\beta,\gamma)\,d\gamma d\beta dy_1 d\alpha dx_1 ds$$

$$+\int_0^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z} p(s,\alpha,\beta,\gamma)\,d\gamma d\beta dy_1 d\alpha dx_1 ds$$

$$-\int_0^t\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} w(s,x,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1 ds$$

$$-\int_0^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{z_0}^{z}\int_{z_0}^{z_1} w(s,\alpha,y,\gamma)\,d\gamma dz_1 d\alpha dx_1 ds$$

$$-\int_0^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1} w(s,\alpha,\beta,z)\,d\beta dy_1\, d\alpha dx_1 ds,$$

$$I_4^{1j}\,(u,v,w,p) = \int_0^t\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u(s,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1 d\alpha ds$$

$$+\int_0^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} v\,(s,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta d\alpha dx_1 ds$$

$$+\int_0^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{z_0}^{y_1}\int_{z_0}^{z_0} w\,(s,\alpha,\beta,\gamma))\,d\gamma d\beta dy_1 d\alpha dx_1 ds,$$

Note that the operators I_k^{1j} corresponds to kth equation of the problem (3.2), $k \in\{1,2,3,4\}$.

Lemma 3.2. *Every solution* $(u,v,w,p) \in \left(C^1([0,1],{C_0}^2(D_j))\right)^4$ *of the system*

$$I_1^{1j}(u,v,w,p) = 0$$

$$I_2^{1j}(u,v,w,p) = 0 \tag{3.3}$$

$$I_3^{1j}(u,v,w,p) = 0$$

$$I_4^{1j}(u,v,w,p) = 0$$

is a solution of the problem (3.2).

Proof. Consider the equation

$$I_1^{1j}(u,v,w,p) = 0 \quad \text{for} \quad (t,x,y,z) \in [0,1] \times D_j.$$

We differentiate it with respect to t and we get

$$0 = \int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u_t\,(t,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1 d\alpha dx_1$$

$$+ \int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z} u^2(t,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1\, d\alpha$$

$$+ \int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{z_0}^{z}\int_{z_0}^{z} u(t,\alpha,\beta,\gamma)\,v(t,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta d\alpha dx_1$$

$$+ \int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z_1} u(t,\alpha,\beta,\gamma)\,w(t,\alpha,\beta,\gamma)\,d\gamma d\beta dy_1 d\alpha dx_1$$

$$+ \int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} p(t,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1\, d\alpha$$

$$- \int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u(t,x,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1$$

$$- \int_{x_0}^{x}\int_{x_0}^{x_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u(t,\alpha,y,\gamma)\,d\gamma dz_1 d\alpha dx_1$$

$$- \int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1} u(t,\alpha,\beta,z)\,d\beta dy_1\, d\alpha dx_1\,, \quad (t,x,y,z) \in [0,1] \times D_j.$$

We differentiate the last equality with respect to x and we obtain

$$0 = \int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u_t\,(t,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1 d\alpha$$

$$+ \int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u^2(t,x,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1$$

$$+ \int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z}\int_{z_0}^{z_1} u(t,\alpha,\beta,\gamma)\,v(t,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta d\alpha$$

$$+\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}u(t,\alpha,\beta,\gamma)\,w(t,\alpha,\beta,\gamma)\,d\gamma d\beta dy_1 d\alpha$$

$$+\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1}p(t,x,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1$$

$$-\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1}u_x(t,x,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1$$

$$-\int_{x_0}^{x}\int_{z_0}^{z}\int_{z_0}^{z_1}u(t,\alpha,y,\gamma)\,d\gamma dz_1 d\alpha$$

$$-\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}u(t,\alpha,\beta,z)\,d\beta dy_1\,d\alpha,\ \ (t,x,y,z)\in[0,1]\times D_j.$$

Again we differentiate with respect to *x* and we get

$$0=\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1}u_t\,(t,x,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1$$

$$+\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1}(\ u^2\,(t,x,\beta,\gamma))_x\,d\gamma dz_1 d\beta dy_1$$

$$+\int_{y_0}^{y}\int_{z_0}^{z}\int_{z_0}^{z_1}u(t,x,\beta,\gamma)\,v(t,x,\beta,\gamma)\,d\gamma dz_1 d\beta$$

$$+\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}u(t,x,\beta,\gamma)\,w(t,x,\beta,\gamma)\,d\gamma d\beta dy_1$$

$$+\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1}p_x(t,x,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1$$

$$-\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1}u_{xx}(t,x,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1$$

$$-\int_{z_0}^{z}\int_{z_0}^{z_1}u(t,x,y,\gamma)\,d\gamma dz_1-\int_{y_0}^{y}\int_{y_0}^{y_1}u(t,x,\beta,z)\,d\beta dy_1,\ \ (t,x,y,z)\in[0,1]\times D_j.$$

Now we differentiate twice the last equation with respect to *y* and we find

$$0=\int_{z_0}^{z}\int_{z_0}^{z_1}u_t\,(t,x,y,\gamma)\,d\gamma dz_1+\int_{z_0}^{z}\int_{z_0}^{z_1}(\ u^2\,(t,x,y,\gamma))_x\,d\gamma dz_1$$

$$+\int_{z_0}^{z}\int_{z_0}^{z_1}(u(t,x,y,\gamma)v(t,x,y,\gamma))_y\,d\gamma dz_1$$

$$+\int_{z_0}^{z}u(t,x,y,\gamma)\,w(t,x,y,\gamma)\,d\gamma+\int_{z_0}^{z}\int_{z_0}^{z_1}p_x(t,x,y,\gamma)\,d\gamma dz_1$$

$$- \int_{z_0}^{z} \int_{z_0}^{z_1} u_{yy}(t, x, y, \gamma)\, d\gamma dz_1 \; - \; u(t, x, y, z)$$

$$- \int_{z_0}^{z} \int_{z_0}^{z_1} u_{xx}(t, x, y, \gamma)\, d\gamma dz_1, \; (t,x,y,z) \in [0,1] \times D_j.$$

We differentiate twice with respect to z the last equation and we get

$$\begin{aligned} 0 \;\; &= u_t(t,x,y,z) + (u^2(t,x,y,z))_x + (u(t,x,y,z)v(t,x,y,z))_y \\ &+ (u(t,x,y,z)w(t,x,y,z))_z + p_x(t,x,y,z) \\ &- u_{xx}(t,x,y,z) - u_{yy}(t,x,y,z) - u_{zz}(t,x,y,z), \; (t,x,y,z) \in [0,1] \times D_j, \end{aligned}$$

i.e., we get the first equation of the system (3.2).

As in above, after we differentiate with respect to t and differentiate twice with respect to x, y and z the equations

$I_2^{1j}(u,v,w,p) = 0$, $I_3^{1j}(u,v,w,p) = 0$, $I_4^{1j}(u,v,w,p) = 0$

we get the second, third and fourth equation of (3.2), respectively.
After we put t=0 in $I_1^{1j} = 0$, we obtain

$$0 = \int_{x_0}^{x} \int_{x_0}^{x_1} \int_{y_0}^{y} \int_{y_0}^{y_1} \int_{z_0}^{z} \int_{z_0}^{z_1} (u(0, \alpha, \beta, \gamma) - \; u_0(\alpha, \beta, \gamma))\, d\gamma dz_1 d\beta dy_1 d\alpha dx_1$$

which we differentiate twice in x, y and z and we find

$$u(0,x,y,z) = u_0(x,y,z) \qquad \text{in} \qquad D_j.$$

After we put t=0 in $I_2^{1j} = 0$ and differentiate twice in x, y and z the equation

$I_2^{1j} = 0$, we obtain

$$v(0,x,y,z) = v_0(x,y,z) \qquad \text{in} \qquad D_j.$$

After we put t=0 in $I_3^{1j} = 0$ and differentiate twice in x, y and z the

equation $I_3^{1j} = 0$, we obtain

$$w(0,x,y,z) = w_0(x,y,z) \qquad \text{in} \qquad D_j.$$

This completes the proof.
The proof of the existence result is based on theorem 2.2 .
Let $\tilde{\tilde{\tilde{x}}}^{1}$ be the set of all equicontinuous families of functions of the space

$$\{g \in C^1([0,1], {C_0}^2(D_j)) : supp_{(x,y,z)}\, g \subset D_{jj}\}$$

with respect to the norm

$||f|| = \max \{\max_{t \in [0,1],(x,y,z)\in D_j} |f(t,x,y,z)|,$

$\max_{t\in[0,1],(x,y,z)\in D_j} |f_t(t,x,y,z)|,$

$\max_{t\in[0,1],(x,y,z)\in D_j} |f_x(t,x,y,z)|,$ $\max_{t\in[0,1],(x,y,z)\in D_j} |f_{xx}(t,x,y,z)|,$

$\max_{t\in[0,1],(x,y,z)\in D_j} |f_y(t,x,y,z)|,$ $\max_{t\in[0,1],(x,y,z)\in D_j} |f_{yy}(t,x,y,z)|,$

$\max_{t\in[0,1],(x,y,z)\in D_j} |f_z(t,x,y,z)|,$ $\max_{t\in[0,1],(x,y,z)\in D_j} |f_{zz}(t,x,y,z)|\}$,

and $\tilde{\tilde{X}}^1 = \tilde{\tilde{\tilde{x}}}^1 \cup \{u_0, v_0, w_0\}$, $\tilde{X}^1 = \overline{\tilde{\tilde{X}}^1}$, i.e., $\tilde{X}^1$ is the completion of $\tilde{\tilde{X}}^1$, and

$$x^1 = \{f \in \tilde{x}^1 : ||f|| \leq M_{1j} := \frac{1}{2^{\frac{j}{1}}\sqrt{\mu(D_j)}}\}$$

Here μ (D_j) is the measure of the set D_j. Let also,

$$N_{1j} := \max\{max_{D_j} |u_0|, max_{D_j} |v_0|, max_{D_j} |w_0|\}$$

We take $\epsilon > 0$ so that

$\epsilon\ (\ 3M_{1j}^2 + \ 6M_{1j} + \ N_{1j}\ )\ \ (\mu(\ D_j\ ))^{\ 2} \leq \ M_{1j}$.

We set

$$Y^1 = \{f \in \tilde{x}^1 : ||f|| \leq (1+\epsilon)\ M_{1j}\}$$

By the construction of X^1 and Y^1, We have that X^1 is a compact subset of $Y^1$and Y^1 is a compact subset of $C^1((0,1], C_0{}^2(D_j))$.

For $u,v,w,p \in Y^1$ we define the operators

S_1^{1j} $(u,v,w,p) = -\in u + \in I_1^{1j}$, T_1^{1j} $(u,v,w,p) = (1+\epsilon)u$,

S_2^{1j} $(u,v,w,p) = -\in v + \in I_2^{1j}$, T_2^{1j} $(u,v,w,p) = (1+\epsilon)v$,

S_3^{1j} $(u,v,w,p) = -\in w + \in I_3^{1j}$, T_3^{1j} $(u,v,w,p) = (1+\epsilon)w$,

S_4^{1j} $(u,v,w,p) = -\in p + \in I_4^{1j}$, T_4^{1j} $(u,v,w,p) = (1+\epsilon)p$,

$S^{1j} = (S_1^{1j}, S_2^{1j}, S_3^{1j}, S_4^{1j})$, $T^{1j} = (T_1^{1j}, T_2^{1j}, T_3^{1j}, T_4^{1j})$.

For $u,v,w,p \in X^1$ we have that

$$|| I_1^{1j}(u,v,w,p)|| \leq (||u|| + ||u||^2 + ||u||.||v|| + ||u||.||w|| + ||p|| + ||u|| + ||u|| + ||u|| + N_{1j}) (\mu(D_j))^2$$

$$\leq (3M_{1j}^2 + 5M_{1j} + N_{1j}) (\mu(D_j))^2.$$

Therefore, using our choice of $\in$

$$|| S_1^{1j} || \leq \in ||u|| + \in (3M_{1j}^2 + 5M_{1j} + N_{1j}) (\mu(D_j))^2$$

$$\leq \in M_{1j} + \in (3M_{1j}^2 + 5M_{1j} + N_{1j}) (\mu(D_j))^2$$

$$\leq (1+\in) M_{1j}.$$

As in above we have

$$|| S_2^{1j} || \leq \in ||v|| + \in (3M_{1j}^2 + 5M_{1j} + N_{1j}) (\mu(D_j))^2$$

$$\leq \in M_{1j} + \in (3M_{1j}^2 + 5M_{1j} + N_{1j}) \mu D_j$$

$$\leq (1+\in) M_{1j}.$$

$$|| S_3^{1j} || \leq \in ||w|| + (\epsilon\, 3M_{1j}^2 + 5M_{1j} + N_{1j}) (\mu(D_j))^2$$

$$\leq \in M_{1j} + (\epsilon\, 3M_{1j}^2 + 5M_{1j} + N_{1j}) (\mu(D_j))^2$$

$$\leq (1+\in) M_{1j},$$

$$|| S_4^{1j} || \leq \in ||p|| + 3\epsilon M_{1j} (\mu(D_j))^2$$

$$\leq \in M_{1j} + M_{1j}$$

$$= (1+\in) M_{1j}.$$

Therefore, for $(u,v,w,p) \in X^1$ we have that

$$S_1^{1j} (u,v,w,p) \in Y^1, \ i = 1,2,3,4.$$

Then

$$S^{1j} : X^1 \times X^1 \times X^1 \times X^1 \to Y^1 \times Y^1 \times Y^1 \times Y^1$$

and it is continuous.

The operator

$$T^{1j} : X^1 \times X^1 \times X^1 \times X^1 \rightarrow Y^1 \times Y^1 \times Y^1 \times Y^1$$

is an expansive operator with constant $1+\in > 1$ and if $(u,v,w,p) \in Y^1 \times Y^1 \times Y^1 \times Y^1$, then

$$\left(\frac{1}{1+\in}u, \frac{1}{1+\in}v, \frac{1}{1+\in}w, \frac{1}{1+\in}p\right) \in X^1 \times X^1 \times X^1 \times X^1,$$

and

$$\Big(T_1^{1j}\left(\frac{1}{1+\in}u, \frac{1}{1+\in}v, \frac{1}{1+\in}w, \frac{1}{1+\in}p\right),\ T_2^{1j}\left(\frac{1}{1+\in}u, \frac{1}{1+\in}v, \frac{1}{1+\in}w, \frac{1}{1+\in}p\right)$$

$$T_3^{1j}\left(\frac{1}{1+\in}u, \frac{1}{1+\in}v, \frac{1}{1+\in}w, \frac{1}{1+\in}p\right),\ T_4^{1j}\left(\frac{1}{1+\in}u, \frac{1}{1+\in}v, \frac{1}{1+\in}w, \frac{1}{1+\in}p\right)\Big)$$

$$= (u,v,w,p).$$

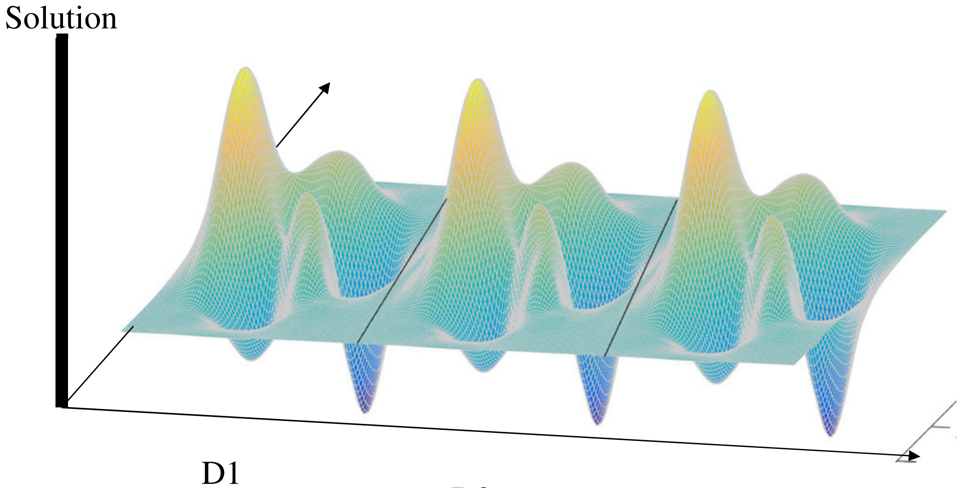
Consequently $T^{1j} : X^1 \times X^1 \times X^1 \times X^1 \rightarrow Y^1 \times Y^1 \times Y^1 \times Y^1$ is onto.

From here and from Theorem 2.3, it follows that the operator $T^{1j} + S^{1j}$ has a fixed point (u_1, v_1, w_1, p_1) in $X^1 \times X^1 \times X^1 \times X^1$. For it we have

$$\begin{cases} T_1^{1j}(u_1, v_1\ w_1, p_1) + S_1^{1j}(u_1, v_1\ w_1, p_1) = u_1 \\ T_2^{1j}(u_1, v_1\ w_1, p_1) + S_2^{1j}(u_1, v_1\ w_1, p_1) = v_1 \\ T_3^{1j}(u_1, v_1\ w_1, p_1) + S_3^{1j}(u_1, v_1\ w_1, p_1) = w_1 \\ T_4^{1j}(u_1, v_1\ w_1, p_1) + S_4^{1j}(u_1, v_1\ w_1, p_1) = p_1 \end{cases}$$

or

$$\begin{cases} (1+\in)u_1 - \in u_1 + I_1^{1j}(u_1, v_1\ w_1, p_1) = u_1 \\ (1+\in)v_1 - \in v_1 + I_2^{1j}(u_1, v_1\ w_1, p_1) = v_1 \\ (1+\in)w_1 - \in w_1 + I_3^{1j}(u_1, v_1\ w_1, p_1) = w_1 \\ (1+\in)p_1 - \in p_1 + I_4^{1j}(u_1, v_1\ w_1, p_1) = p_1 \end{cases}$$



Disjointed sets of R^3

Figure 1: A sketch of the division of $R^3$to disjointed subsets D1,D2,D3 etc, that illustrates how the proof of existence steps are done.

Whereupon

$$I_1^{1j}(u_1, v_1\ w_1, p_1) = 0, \quad I_2^{1j}(u_1, v_1\ w_1, p_1) = 0$$

$$I_3^{1j}(u_1, v_1\ w_1, p_1) = 0, \quad I_4^{1j}(u_1, v_1\ w_1, p_1) = 0$$

Hence and Lemma 3.2 we obtain that (u_1,v_1,w_1,p_1) is a solution of the system (3.2) for which $u_1,v_1,w_1,p_1 \in C^1([0,1], C_0{}^2(D_j))$.

Remark 3.3. If we assume that

$$u_1(x,y,z) = u_0(x,y,z),$$

$$v_1(x,y,z) = v_0(x,y,z),$$

$$w_1(x,y,z) = w_0(x,y,z), t \in [0,1], (x,y,z) \in D_j$$

then using $I_l^{ij}(u_1, v_1\ w_1, p)(t, x, y, z) = 0, t \in [0,1], (x,y,z) \in D_j,$

$l = 1,2,3$ *we get*

$$t\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1}(u_0(\alpha,\beta,\gamma))^2\, d\gamma dz_1 d\beta dy_1 d\alpha dx_1$$

$$+ t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{z_0}^{z}\int_{z_0}^{z_1} u_0(\alpha,\beta,\gamma)\, v_0(\alpha,\beta,\gamma))\, d\gamma dz_1 d\beta d\alpha dx_1$$

$$+ t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z_1} u_0(\alpha,\beta,\gamma)\, w_0(\alpha,\beta,\gamma)\, d\gamma d\beta dy_1 d\alpha dx_1$$

$$+ \int_{0}^{t}\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} p_1(\alpha,\beta,\gamma)\, d\gamma dz_1 d\beta dy_1 d\alpha ds$$

$$-t\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u_0(x,\beta,\gamma)\; d\gamma dz_1 d\beta dy_1$$

$$- t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1} u_0(\alpha,\beta,z)\; d\beta dy_1\, d\alpha dx_1$$

$$- t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u_0(\alpha,y,\gamma)\; d\gamma dz_1\, d\alpha dx_1 = o\,,\ t\in[0,1],\ (x,y,z)\in D_j, \qquad (3.4)$$

$$t\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u_0(\alpha,\beta,\gamma)\, v_0(\alpha,\beta,\gamma)\, d\gamma dz_1 d\beta dy_1 d\alpha dx_1$$

$$+ t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{z_0}^{z}\int_{z_0}^{z_1} (\, v_0(\alpha,\beta,\gamma))^2\; d\gamma dz_1 d\beta d\alpha dx_1$$

$$+ t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z_1} v_0(\alpha,\beta,\gamma)\, w_0(\alpha,\beta,\gamma))\, d\gamma d\beta dy_1 d\alpha dx_1$$

$$+ \int_{0}^{t}\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{z_0}^{z}\int_{z_0}^{z_1} p_1(s,\alpha,\beta,\gamma)\, d\gamma dz_1 d\beta d\alpha dx_1 ds$$

$$-t\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} v_0(x,\beta,\gamma)\; d\gamma dz_1 d\beta dy_1$$

$$- t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1} v_0(\alpha,\beta,z)\; d\beta dy_1\, d\alpha dx_1$$

$$- t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{z_0}^{z}\int_{z_0}^{z_1} v_0(\alpha,y,\gamma)\; d\gamma dz_1\, d\alpha dx_1 = 0\,,\ t\in[0,1],\ (x,y,z)\in D_j, \qquad (3.5)$$

$$t\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u_0(\alpha,\beta,\gamma)\, w_0(\alpha,\beta,\gamma)\, d\gamma dz_1 d\beta dy_1 d\alpha dx_1$$

$$
\begin{aligned}
&+ t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{z_0}^{z}\int_{z_0}^{z_1} w_0\,(\alpha,\beta,\gamma)\, v_0\,(\alpha,\beta,\gamma)\, d\gamma dz_1 d\beta d\alpha dx_1 \\
&+ t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z_1} (\, w_0\,(\alpha,\beta,\gamma))^2\; d\gamma d\beta dy_1 d\alpha dx_1 \\
&+ \int_{0}^{t}\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z_1} p_1\,(s,\alpha,\beta,\gamma)\, d\gamma dd\beta dy_1 d\alpha dx_1 ds \\
&-t\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} w_0\,(x,\beta,\gamma)\;\; d\gamma dz_1 d\beta dy_1 \\
&- t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1} w_0\,(\alpha,\beta,z)\;\; d\beta dy_1\, d\alpha dx_1 \\
&- t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1} w_0\,(\alpha,y,z)\;\; d\gamma dz_1\, d\alpha dx_1 = 0\; t\; \in\; [0,1], (\mathrm{x,y,z})\; \in\; Dj,
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
&t\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u_0\,(\alpha,\beta,\gamma)\, d\gamma dz_1 d\beta dy_1 d\alpha \\
&+ t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{z_0}^{y_1}\int_{z_0}^{z} u_0\,(\alpha,\beta,\gamma)\, d\gamma d\beta dy_1 d\alpha dx_1 \\
&+ t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z_1} u_0\,(\alpha,\beta,\gamma)\, d\gamma dz_1 d\beta d\alpha dx_1 = 0, t \in [0,1], (\mathrm{x,y,z}) \in\; Dj,
\end{aligned} \tag{3.7}
$$

$If\ the\ initial\ functions\ \ u_0\,,\ v_0\,,\ w_0\ \ and\ \ p_0\ \ do\ not\ satisfy\ (3.4), (3.5), (3.6), (3.7), then$

$$(\,u_0\,,\ v_0\,,\ w_0\,) \not\equiv (\,u_1\,,\ v_1\,,\ w_1\,) \text{ on } [\mathrm{o},1] \times Dj$$

Step 2. Now we consider the problem

$$
\begin{cases}
u_t + (u^2)_x + (uv)_y + (uw)_z + p_x - u_{xx} - u_{yy} - u_{zz} = 0 \\
v_t + (uv)_x + (v^2)_y + (vw)_z + p_y - v_{xx} - v_{yy} - v_{zz} = 0 \\
w_t + (uw)_x + (vw)_y + (w^2)_z + p_z - w_{xx} - w_{yy} - w_{zz} = 0 \\
u_x + v_y + w_z = 0 \quad \text{in } (1,2] \times \; D_j\,, \\
u(1,x,y,z) = \; u_1\,(1,x,y,z)\,, \quad v(1,x,y,z) = \; v_1\,(1,x,y,z), \\
w\,(1,x,y,z) = \; w_1\,(1,x,y,z) \quad in \quad D_j.
\end{cases} \tag{3.8}
$$

We will prove that the problem (3.8) has a solution (*u,v,w,p*) such that $u,v,w,p \in C^1([1,2], C_0{}^2(D_j))$.

Let $(x_0,y_0,z_0) \in D_j$ be arbitrarily chosen.

For $(u,v,w,p) \in (C^1([1,2], C_0{}^2(D_j)))^4$ we define

$$I_1^{2j}(u,v,w,p) = \int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1}(u(t,\alpha,\beta,\gamma) - u_1(1,\alpha,\beta,\gamma))\,d\gamma dz_1 d\beta dy_1 d\alpha dx_1$$

$$+\int_1^t\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u^2(s,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1 d\alpha ds$$

$$+\int_1^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{z_0}^{z}\int_{z_0}^{z_1} u(s,\alpha,\beta,\gamma)v(s,\alpha,\beta,\gamma))\,d\gamma dz_1 d\beta dy_1 d\alpha dx_1 ds$$

$$+\int_1^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z} u(s,\alpha,\beta,\gamma)w(s,\alpha,\beta,\gamma))\,d\gamma d\beta dy_1 d\alpha dx_1 ds$$

$$+\int_1^t\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} p(s,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1 d\alpha ds$$

$$-\int_1^t\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u(s,x,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1 ds$$

$$-\int_1^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u(s,\alpha,y,\gamma)\,d\gamma dz_1 d\alpha dx_1 ds$$

$$-\int_1^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{z_0}^{y}\int_{z_0}^{y_1} u(s,\alpha,\beta,z)\,d\beta dy_1\, d\alpha dx_1 ds$$

$$I_2^{2j}(u,v,w,p) =$$

$$\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1}(v(t,\alpha,\beta,\gamma) - v_1(1,\alpha,\beta,\gamma))\,d\gamma dz_1 d\beta dy_1 d\gamma dx_1$$

$$+\int_1^t\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u\,(s,\alpha,\beta,\gamma)v\,(s,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1 d\alpha ds$$

$$+\int_1^t\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{z_0}^{z}\int_{z_0}^{z_1} v^2(s,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta d\alpha dx_1 ds$$

$$+\int_1^t \int_{x_0}^{x} \int_{x_0}^{x_1} \int_{y_0}^{y} \int_{y_0}^{y_1} \int_{z_0}^{z} v(s,\alpha,\beta,\gamma)w(s,\alpha,\beta,\gamma))\, d\gamma d\beta dy_1 d\alpha dx_1 ds$$

$$+\int_1^t \int_{x_0}^{x} \int_{x_0}^{x_1} \int_{y_0}^{y} \int_{z_0}^{z} \int_{z_0}^{z_1} p(s,\alpha,\beta,\gamma)\, d\gamma dz_1 d\beta d\alpha dx_1 ds$$

$$-\int_1^t \int_{y_0}^{y} \int_{y_0}^{y_1} \int_{z_0}^{z} \int_{z_0}^{z_1} v(s,x,\beta,\gamma)\, d\gamma dz_1 d\beta dy_1 ds$$

$$-\int_1^t \int_{x_0}^{x} \int_{x_0}^{x_1} \int_{z_0}^{z} \int_{z_0}^{z_1} v(s,\alpha,y,\gamma)\, d\gamma dz_1 d\alpha dx_1 ds$$

$$-\int_1^t \int_{x_0}^{x} \int_{x_0}^{x_1} \int_{y_0}^{y} \int_{y_0}^{y_1} v(s,\alpha,\beta,z)\, d\beta dy_1\, d\alpha dx_1 ds$$

$I_3^{2j}\,(u,v,w,p) =$

$$\int_{x_0}^{x} \int_{x_0}^{x_1} \int_{y_0}^{y} \int_{y_0}^{y_1} \int_{z_0}^{z} \int_{z_0}^{z_1} (w(t,\alpha,\beta,\gamma) - w_1(1,\alpha,\beta,\gamma))\, d\gamma dz_1 d\beta dy_1 d\gamma dx_1$$

$$+\int_1^t \int_{x_0}^{x} \int_{y_0}^{y} \int_{y_0}^{y_1} \int_{z_0}^{z} \int_{z_0}^{z_1} u\,(s,\alpha,\beta,\gamma)w\,(s,\alpha,\beta,\gamma)\, d\gamma dz_1 d\beta dy_1 d\alpha ds$$

$$+\int_1^t \int_{x_0}^{x} \int_{x_0}^{x} \int_{y_0}^{y} \int_{z_0}^{z} \int_{z_0}^{z_1} v\,(s,\alpha,\beta,\gamma)w\,(s,\alpha,\beta,\gamma)\, d\gamma dz_1 d\beta d\alpha dx_1 ds$$

$$+\int_1^t \int_{x_0}^{x} \int_{x_0}^{x_1} \int_{y_0}^{y} \int_{y_0}^{y_1} \int_{z_0}^{z} w^2(s,\alpha,\beta,\gamma)\, d\gamma d\beta dy_1 d\alpha dx_1 ds$$

$$+\int_1^t \int_{x_0}^{x} \int_{x_0}^{x_1} \int_{y_0}^{y} \int_{y_0}^{y_1} \int_{z_0}^{z} p(s,\alpha,\beta,\gamma)\, d\gamma d\beta dy_1 d\alpha dx_1 ds$$

$$-\int_1^t \int_{y_0}^{y} \int_{y_0}^{y_1} \int_{z_0}^{z} \int_{z_0}^{z_1} w(s,x,\beta,\gamma)\, d\gamma dz_1 d\beta dy_1 ds$$

$$-\int_1^t \int_{x_0}^{x} \int_{x_0}^{x_1} \int_{z_0}^{z} \int_{z_0}^{z_1} w(s,\alpha,y,\gamma)\, d\gamma dz_1 d\alpha dx_1 ds$$

$$-\int_1^t \int_{x_0}^{x} \int_{x_0}^{x_1} \int_{y_0}^{y} \int_{y_0}^{y_1} w(s,\alpha,\beta,z)\, d\beta dy_1\, d\alpha dx_1 ds$$

$I_4^{2j}\ (u,v,w,p) =$

$$\int_1^t \int_{x_0}^x \int_{y_0}^y \int_{y_0}^{y_1} \int_{z_0}^z \int_{z_0}^{z_1} u\ (s,\alpha,\beta,\gamma)\ d\gamma dz_1 d\beta dy_1 d\alpha ds$$

$$+\int_1^t \int_{x_0}^x \int_{x_0}^{x_1} \int_{y_0}^y \int_{z_0}^z \int_{z_0}^{z_1} v\ (s,\alpha,\beta,\gamma)\ d\gamma dz_1 d\beta d\alpha dx_1 ds$$

$$+\int_1^t \int_{x_0}^x \int_{x_0}^x \int_{y_0}^y \int_{z_0}^z \int_{z_0}^{z_1} w(s,\alpha,\beta,\gamma)\ d\gamma d\beta dy_1 d\alpha dx_1 ds.$$

We note that after we differentiate with respect to *t* and twice with respect to *x*, *y* and *z* the system

$$I_1^{2j}\ (u,v,w,p) = 0, \qquad I_2^{2j}\ (u,v,w,p) = 0, \qquad (3.9)$$
$$I_3^{2j}\ (u,v,w,p) = 0, \qquad I_4^{2j}\ (u,v,w,p) = 0,$$

we get the system (3.8). After we put t=1 in $I_1^{2j} = 0$ and differentiate twice in *x*, *y* and *z* the equation $I_1^{2j} = 0$ we obtain

$$u(1,x,y,z) = u_1(1,x,y,z) \qquad \text{in} \qquad D_j.$$

After we put t=1 in $I_2^{2j} = 0$ and differentiate twice in *x*, *y* and *z* the equation

$I_2^{2j} = 0$ we obtain

$$v(1,x,y,z) = v_1(1,x,y,z) \qquad \text{in} \qquad D_j.$$

After we put t=1 in $I_3^{2j} = 0$ and differentiate twice in *x*, *y* and *z* the equation $I_3^{2j} = 0$ we obtain

$$w(1,x,y,z) = w_1(1,x,y,z) \qquad \text{in} \qquad D_j.$$

Consequently every solution $(u,v,w,p) \in \left(C^1([1,2], {C_0}^2(D_j))\right)^4$ of (3.9) is a solution of the problem (3.8).

Let $\tilde{\tilde{\tilde{x}}}^{\,2}$ be a equicontinuous family of functions of the space

$$\{g \in C^1([1,2], {C_0}^2(D_j)),\ supp_{x,y,z}\ g \subset D_{jj}\}$$

with respect to the norm

$||f|| = \max \{ \max_{t\in[1,2],(x,y,z)\in D_j} |f(t,x,y,z)|,$

$\max_{t\in[1,2],(x,y,z)\in D} |f_t(t,x,y,z)|$,

$\max_{t\in[1,2],(x,y,z)\in D_j} |f_x(t,x,y,z)|$, $\max_{t\in[1,2],(x,y,z)\in D_j} |f_{xx}(t,x,y,z)|$,

$\max_{t\in[1,2],(x,y,z)\in D_j} |f_y(t,x,y,z)|$, $\max_{t\in[1,2],(x,y,z)\in D_j} |f_{yy}(t,x,y,z)|$,

$\max_{t\in[1,2],(x,y,z)\in D_j} |f_z(t,x,y,z)|$, $\max_{t\in[1,2],(x,y,z)\in D_j} |f_{zz}(t,x,y,z)|$,

$f \in \tilde{X}^2$,

and

$\tilde{\tilde{X}}^2 = \tilde{\tilde{\tilde{x}}}^2 \cup \{ u_1(t,x,y,z),\ v_1((t,x,y,z),\ w_1((t,x,y,z)\}$,

$\tilde{X}^2 = \overline{\tilde{\tilde{X}}^2}$, i.e., $\tilde{X}^2$is the completion of $\tilde{\tilde{X}}^2$, and

$$X^2 = \{f \ \in \tilde{x}^2 : ||f|| \ \le \ M_j = \frac{1}{2^{\frac{j}{2}}\sqrt{\mu(D_j)}}\}$$

And

$$Y^2 = \{f \ \in \tilde{X}^2 : ||f|| \ \le \ (1+\epsilon)\, M_{1j}\}$$

Note that X^2 is a compact subset of Y^2.

For $u,v,w,p \in Y^2$ we define the operators

S_1^{2j} (u,v,w,p) = $-\in u + \in I_1^{2j}$, T_1^{2j} $(u,v,w,p) = (1+\epsilon)u$,

S_2^{2j} (u,v,w,p) = $-\in v + \in I_2^{2j}$, T_2^{2j} $(u,v,w,p) = (1+\epsilon)v$,

S_3^{2j} (u,v,w,p) = $-\in w + \in I_3^{2j}$, T_3^{2j} $(u,v,w,p) = (1+\epsilon)w$,

S_4^{2j} (u,v,w,p) = $-\in p + \in I_4^{1j}$, T_4^{2j} $(u,v,w,p) = (1+\epsilon)p$,

$S^{2j} = (S_1^{2j}, S_2^{2j},\ S_3^{2j},\ S_4^{2j})$, $T^{2j} = (T_1^{2j}, T_2^{2j}, T_3^{2j}, T_4^{2j})$.

For $u,v,w,p \in X^2$ we have that $S_i^{2j}(u,v,w,p) \in Y^2$, i = 1,2,3,4, i.e.,

$$S^{2j} :: X^2 \times X^2 \times X^2 \times X^2 \to Y^2 \times Y^2 \times Y^2 \times Y^2$$

and it is continuous.

The operator

$$T^{2j} : X^2 \times X^2 \times X^2 \times X^2 \to Y^2 \times Y^2 \times Y^2 \times Y^2$$

is an expansive operator with constant 1+ ∈ > 1and if $(u,v,w,p) \in Y^2 \times Y^2 \times Y^2 \times Y^2$, then

$$\left(\frac{1}{1+\in}u, \frac{1}{1+\in}v, \frac{1}{1+\in}w, \frac{1}{1+\in}p\right) \in X^2 \times X^2 \times X^2 \times X^2,$$

and

$$\Big(T_1^{2j}\left(\frac{1}{1+\in}u, \frac{1}{1+\in}v, \frac{1}{1+\in}w, \frac{1}{1+\in}p\right), T_2^{2j}\left(\frac{1}{1+\in}u, \frac{1}{1+\in}v, \frac{1}{1+\in}w, \frac{1}{1+\in}p\right)$$

$$T_3^{2j}\left(\frac{1}{1+\in}u, \frac{1}{1+\in}v, \frac{1}{1+\in}w, \frac{1}{1+\in}p\right), T_4^{2j}\left(\frac{1}{1+\in}u, \frac{1}{1+\in}v, \frac{1}{1+\in}w, \frac{1}{1+\in}p\right)\Big)$$

$$= (u,v,w,p).$$

Consequently $T^{2j} : X^2 \times X^2 \times X^2 \times X^2 \to Y^2 \times Y^2 \times Y^2 \times Y^2$ is onto.

From here and from Theorem 2.3, it follows that the operator $T^{2j} + S^{2j}$ has a fixed point (u_2,v_2,w_2,p_2) in $X^2 \times X^2 \times X^2 \times X^2$. For it we have

$$\begin{cases} T_1^{2j}(u_2, v_2, w_2, p_2) + S_1^{2j}(u_2, v_2, w_2, p_2) = u_2 \\ T_2^{2j}(u_2, v_2, w_2, p_2) + S_2^{2j}(u_2, v_2, w_2, p_2) = v_2 \\ T_3^{2j}(u_2, v_2, w_2, p_2) + S_3^{2j}(u_2, v_2, w_2, p_2) = w_2 \\ T_4^{2j}(u_2, v_2, w_2, p_2) + S_4^{2j}(u_2, v_2, w_2, p_2) = p_2 \end{cases}$$

Or,

$$\begin{cases} (1+\in)\, u_2 - \in\, u_2 + I_1^{2j}(u_2, v_2, w_2, p_2) = u_2 \\ (1+\in)\, v_2 - \in\, v_2 + I_2^{2j}(u_2, v_2, w_2, p_2) = v_2 \\ (1+\in)\, w_2 - \in\, w_2 + I_3^{2j}(u_2, v_2, w_2, p_2) = w_2 \\ (1+\in)\, p_2 - \in\, p_2 + I_4^{2j}(u_2, v_2, w_2, p_2) = p_2, \end{cases}$$

Whereupon

$$I_1^{2j}(u_2, v_2, w_2, p_2) = 0, \quad I_2^{2j}(u_2, v_2, w_2, p_2) = 0$$

$$I_3^{2j}(u_2, v_2, w_2, p_2) = 0, \quad I_4^{2j}(u_2, v_2, w_2, p_2) = 0$$

Therefore (u_2,v_2,w_2,p_2) is a solution of the system (3.8) for which $u_2, v_2, w_2, p_2 \in C^1([1,2], C_0{}^2(D_j))$.

We note that

$$u_1(1,x,y,z) = u_2(1,x,y,z),$$

$$v_1(1,x,y,z) = v_2(1,x,y,z),$$

$$w_1(1,x,y,z) = w_2(1,x,y,z),$$

$$p_1(1,x,y,z) = p_2(1,x,y,z),$$

whereupon

$$u_{1x}(1,x,y,z) = u_{2x}(1,x,y,z),$$

$$v_{1x}(1,x,y,z) = v_{2x}(1,x,y,z),$$

$$w_{1x}(1,x,y,z) = w_{2x}(1,x,y,z),$$

$$p_{1x}(1,x,y,z) = p_{2x}(1,x,y,z),$$

$$u_{1y}(1,x,y,z) = u_{2y}(1,x,y,z),$$

$$v_{1y}(1,x,y,z) = v_{2y}(1,x,y,z),$$

$$w_{1y}(1,x,y,z) = w_{2y}(1,x,y,z),$$

$$p_{1y}(1,x,y,z) = p_{2y}(1,x,y,z),$$

$$u_{1z}(1,x,y,z) = u_{2z}(1,x,y,z),$$

$$v_{1z}(1,x,y,z) = v_{2z}(1,x,y,z),$$

$$w_{1z}(1,x,y,z) = w_{2z}(1,x,y,z),$$

$$p_{1z}(1,x,y,z) = p_{2z}(1,x,y,z),$$

$$u_{1xx}(1,x,y,z) = u_{2xx}(1,x,y,z),$$

$$v_{1xx}(1,x,y,z) = v_{2xx}(1,x,y,z),$$

$$w_{1xx}(1,x,y,z) = w_{2xx}(1,x,y,z),$$

$$p_{1xx}(1,x,y,z) = p_{2xx}(1,x,y,z),$$

$$u_{1yy}(1,x,y,z) = u_{2yy}(1,x,y,z),$$

$$v_{1yy}(1,x,y,z) = v_{2yy}(1,x,y,z),$$

$$w_{1yy}(1,x,y,z) = w_{2yy}(1,x,y,z),$$

$$p_{1yy}(1,x,y,z) = p_{2yy}(1,x,y,z),$$

$$u_{1zz}(1,x,y,z) = u_{2zz}(1,x,y,z),$$

$$v_{1zz}(1,x,y,z) = v_{2zz}(1,x,y,z),$$

$$w_{1zz}(1,x,y,z) = w_{2zz}(1,x,y,z),$$

$$p_{1zz}(1,x,y,z) = p_{2zz}(1,x,y,z).$$

Hence and (2.2), (2.4), we get

$$u_{1t}(1,x,y,z) = u_{2t}(1,x,y,z),$$

$$v_{1t}(1,x,y,z) = v_{2t}(1,x,y,z),$$

$$w_{1t}(1,x,y,z) = w_{2t}(1,x,y,z),$$

$$p_{1t}(1,x,y,z) = p_{2t}(1,x,y,z).$$

Consequently

$(u(t,x,y,z),v(t,x,y,z),w(t,x,y,z),p(t,x,y,z)) =$

$$\begin{cases} (u_1(t,x,y,z), v_1(t,x,y,z), w_1(t,x,y,z), p_1(t,x,y,z)) \in (C^1([0,1], {C_0}^2(Dj)))^4 \\ (u_2(t,x,y,z), v_2(t,x,y,z), w_2(t,x,y,z), p_2(t,x,y,z)) \in (C^1([1,2], {C_0}^2(Dj)))^4 \end{cases}$$

belongs to $(C^1([0,2], {C_0}^2(D_j)))^4$ and it is a solution to the problem

$$\begin{cases} u_t + (u^2)_x + (uv)_y + (uw)_z + p_x - u_{xx} - u_{yy} - u_{zz} = 0 \\ v_t + (uv)_x + (v^2)_y + (vw)_z + p_y - v_{xx} - v_{yy} - v_{zz} = 0 \\ w_t + (uw)_x + (vw)_y + (w^2)_z + p_z - w_{xx} - w_{yy} - w_{zz} = 0 \\ u_x + v_y + w_z = 0 \quad \text{in } (0,2] \times D_j , \\ u(0,x,y,z) = u_0(x,y,z), \quad v(0,x,y,z) = v_0(x,y,z), \\ w(0,x,y,z) = w_0(x,y,z) \quad in \quad D_j. \end{cases}$$

Then we consider the problem

$$\begin{cases} u_t + (u^2)_x + (uv)_y + (uw)_z + p_x - u_{xx} - u_{yy} - u_{zz} = 0 \\ v_t + (uv)_x + (v^2)_y + (vw)_z + p_y - v_{xx} - v_{yy} - v_{zz} = 0 \\ w_t + (uw)_x + (vw)_y + (w^2)_z + p_z - w_{xx} - w_{yy} - w_{zz} = 0 \\ u_x + v_y + w_z = 0 \quad \text{in } (2,3] \times D_j , \\ u(2,x,y,z) = u_2(2,x,y,z), \quad v(2,x,y,z) = v_2(2,x,y,z), \\ w(2,x,y,z) = w_2(2,x,y,z) \quad in \quad D_j \end{cases}$$

and as above we construct a solution

$$(u_3, v_3, w_3, p_3) \in (C^1([2,3], {C_0}^2(D_j)))^4$$

and so on. Consequently

$(u^j(t,x,y,z), v^j(t,x,y,z), w^j(t,x,y,z), p^j(t,x,y,z))$

$$\begin{cases} (u_1(t,x,y,z), v_1(t,x,y,z), w_1(t,x,y,z), p_1(t,x,y,z)) \in (C^2([0,1], {C_0}^2(Dj)))^4 \\ (u_2(t,x,y,z), v_2(t,x,y,z), w_2(t,x,y,z), p_2(t,x,y,z)) \in (C^2([1,2], {C_0}^2(Dj)))^4 \\ (u_3(t,x,y,z), v_3(t,x,y,z), w_3(t,x,y,z), p_3(t,x,y,z)) \in (C^2([2,3], {C_0}^2(Dj)))^4 \\ (u_4(t,x,y,z), v_4(t,x,y,z), w_4(t,x,y,z), p_4(t,x,y,z)) \in (C^2([3,4], {C_0}^2(Dj)))^4 \\ \dots \dots \dots \end{cases}$$

belongs to $(C^1([0,\infty), C_0^2(D_j)))^4$ and it is a solution to the problem (3.2).

Note that
$\operatorname{supp} u^j, \operatorname{supp} v^j, \operatorname{supp} w^j, \operatorname{supp} p^j \subset D_{jj} \subset D_j$ for any $j = 1,2,\ldots.$

and then

$$({D_{txyz}}^{\alpha} u^j, {D_{txyz}}^{\alpha} v^j, {D_{txyz}}^{\alpha} w^j, {D_{txyz}}^{\alpha} p^j)\big|_{\partial D_j}$$

$$= ({D_{txyz}}^{\alpha} u^{j+1}, {D_{txyz}}^{\alpha} v^{j+1}, {D_{txyz}}^{\alpha} w^{j+1}, {D_{txyz}}^{\alpha} p^{j+!})\big|_{\partial D_{j+1}}$$

$= 0$

for any $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \{0,1,\ldots\}$. Also,

$$(u,v,w,p) = \begin{cases} (u^1, v^1, w^1, p^1) \in (C^2([0,\infty), C_0{}^2(D_1)))^4 \\ (u^2, v^2, w^2, p^2) \in (C^2([0,\infty), C_0{}^2(D_2)))^4 \\ (u^3, v^3, w^3, p^3) \in (C^2([0,\infty), C_0{}^2(D_3)))^4 \\ \qquad\qquad \cdots \end{cases}$$

is a solution to the problem (1.1) which belongs to the space $(C^2([0,\infty) \times R^3))^4$. Using the system (1.1) we have that $(u,v,w,p) \in (C^\infty([0,\infty) \times R^3))^4$.

Therefore

supp$u$, supp$v$, supp$w$, supp$p \subset D_{11} \cup D_{22} \cup \ldots$

Also,

$$\int_{R^3} |u(t,x,y,z)|^2\,dx\,dydz = \sum_{j=1}^{\infty} \int_{D_j} |\,u^j(t,x,y,z)|^2\,dx\,dydz \le \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty,$$

$$\int_{R^3} |v(t,x,y,z)|^2\,dx\,dydz = \sum_{j=1}^{\infty} \int_{D_j} |\,v^j(t,x,y,z)|^2\,dx\,dydz \le \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty,$$

$$\int_{R^3} |w(t,x,y,z)|^2\,dx\,dydz = \sum_{j=1}^{\infty} \int_{D_j} |\,w^j(t,x,y,z)|^2\,dx\,dydz \le \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty,$$

$$\int_{R^3} |p(t,x,y,z)|^2\,dx\,dydz = \sum_{j=1}^{\infty} \int_{D_j} |\,p^j(t,x,y,z)|^2\,dx\,dydz \le \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty,$$

This completes the proof

Remark 3.4. *We note that in mth step we have*

$$I_1^{mj}\,(u,v,w,p) = \int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1}(u(t,\alpha,\beta,\gamma) - u_{m-1}(m-1,\alpha,\beta,\gamma))\,d\gamma dz_1 d\beta dy_1 d\alpha dx_1$$

$$+\int_{m-1}^{t}\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u^2(s,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1 d\alpha ds$$
$$+\int_{m-1}^{t}\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{z_0}^{z}\int_{z_0}^{z_1} u(s,\alpha,\beta,\gamma)v(s,\alpha,\beta,\gamma))\,d\gamma dz_1 d\beta d\alpha dx_1 ds$$
$$+\int_{m-1}^{t}\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z} u(s,\alpha,\beta,\gamma)w(s,\alpha,\beta,\gamma))\,d\gamma d\beta dy_1 d\alpha dx_1 ds$$

$$+\int_{m-1}^{t}\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} p(s,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1 d\alpha ds$$

$$-\int_{m-1}^{t}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u(s,x,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1 ds$$

$$-\int_{m-1}^{t}\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u(s,\alpha,y,\gamma)\,d\gamma dz_1 d\alpha dx_1 ds$$

$$-\int_{m-1}^{t}\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{z_0}^{y}\int_{z_0}^{y_1} u(s,\alpha,\beta,z)\,d\beta dy_1\, d\alpha dx_1 ds$$

$$I_2^{mj}\,(u,v,w,p) =$$

$$\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1}(v(t,\alpha,\beta,\gamma) - v_{m-1}(m-1,\alpha,\beta,\gamma))\,d\gamma dz_1 d\beta dy_1 d\alpha dx_1$$

$$+\int_{m-1}^{t}\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u\,(s,\alpha,\beta,\gamma)v\,(s,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1 d\alpha ds$$

$$+\int_{m-1}^{t}\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{z_0}^{z}\int_{z_0}^{z_1} v^2(s,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1 d\alpha dx_1 ds$$

$$+\int_{m-1}^{t}\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z} v(s,\alpha,\beta,\gamma)w\,(s,\alpha,\beta,\gamma))\,d\gamma d\beta dy_1 d\alpha dx_1 ds$$

$$+\int_{m-1}^{t}\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{z_0}^{z}\int_{z_0}^{z_1} p(s,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta d\alpha dx_1 ds$$

$$-\int_{m-1}^{t}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} v(s,x,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1 ds$$

$$-\int_{m-1}^{t}\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{z_0}^{z}\int_{z_0}^{z_1} v(s,\alpha,y,\gamma)\,d\gamma dz_1 d\alpha dx_1 ds$$

$$-\int_{m-1}^{t}\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1} v(s,\alpha,\beta,z)\,d\beta dy_1\, d\alpha dx_1 ds$$

$$I_3^{mj}\,(u,v,w,p) =$$

$$\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1}(w(t,\alpha,\beta,\gamma) - w_{m-1}(m-1,\alpha,\beta,\gamma))\,d\gamma dz_1 d\beta dy_1 d\gamma dx_1$$

$$+\int_{m-1}^{t}\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u\,(s,\alpha,\beta,\gamma)w\,(s,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1 d\alpha ds$$

$$+\int_{m-1}^{t}\int_{x_0}^{x}\int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z}\int_{z_0}^{z_1} v\,(s,\alpha,\beta,\gamma)w\,(s,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta d\alpha dx_1 ds$$

$$+\int_{1}^{t}\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z} w^2(s,\alpha,\beta,\gamma)\,d\gamma d\beta dy_1 d\alpha dx_1 ds$$

$$+\int_{m-1}^{t}\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z} p(s,\alpha,\beta,\gamma)\,d\gamma d\beta dy_1 d\alpha dx_1 ds$$

$$-\int_{m-1}^{t}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} w(s,x,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1 ds$$

$$-\int_{m-1}^{t}\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{z_0}^{z}\int_{z_0}^{z_1} w(s,\alpha,y,\gamma)\,d\gamma dz_1 d\alpha dx_1 ds$$

$$-\int_{m-1}^{t}\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1} w(s,\alpha,\beta,z)\,d\beta dy_1\, d\alpha dx_1 ds$$

$$I_4^{mj}\,(u,v,w,p) =$$

$$\int_{0}^{t}\int_{x_0}^{x}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z}\int_{z_0}^{z_1} u\,(s,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta dy_1 d\alpha ds$$

$$+\int_{m-1}^{t}\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{z_0}^{z}\int_{z_0}^{z_1} v\,(s,\alpha,\beta,\gamma)\,d\gamma dz_1 d\beta d\alpha dx_1 ds$$

$$+\int_{m-1}^{t}\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{y_0}^{y}\int_{y_0}^{y_1}\int_{z_0}^{z} w(s,\alpha,\beta,\gamma)\,d\gamma d\beta dy_1 d\alpha dx_1 ds$$

Here $(x_0,y_0,z_0) \in D_j$, $j \in \{1,2,\ldots\}$.

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