EXISTENCE AND SMOOTHNESS OF SOLUTIONS TO THE 3D NAVIER-STOKES EQUATION – VERSION 2023-12-30

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ABSTRACT. Existence of a solution to the 3D Navier-Stokes equation with a smooth initial fluid velocity profile is proven assuming a scalar pressure function and incompressible flow. Specifically, it is assumed that the absolute value of the initial velocity profile and all of its spatial derivatives approach zero as $1/(|\mathbf{x}| + a)^{\kappa}$ as $|\mathbf{x}| \to \infty$, where κ is a constant such that $3/2 < \kappa \leq 2$, and a is a positive constant. Also, it is taken as a given that the fluid is flowing in free space under the forces of viscosity and scalar pressure gradients only, and that there are no external driving forces.

First, we show that for any smooth velocity profile with the properties described above, there exists a scalar pressure gradient with an absolute value that also approaches zero as $1/(|\mathbf{x}| + a)^{\kappa}$ as $|\mathbf{x}| \to \infty$. This scalar pressure arises from the incompressibility of the fluid, and maintains a zero-divergence fluid velocity over time. We then show that any fluid velocity solution would retain this spatial smoothness profile when propagated in time for as long as the solution exists (ie. remains finite). Next, we conjecture that a blowup solution exists which becomes infinite at some point $\mathbf{x}_b$ and time t_b. Since this is a smooth blowup, however, it can form only if a global maximum in the kinetic energy density $K = \mathbf{u} \cdot \mathbf{u}/2$ (where $\mathbf{u}$ is the fluid velocity) forms at $\mathbf{x}_b$ and a time t_m such that $t_m < t_b$. If such a global maximum in K does not occur before the blowup, then the blowup will occur first at another point $\mathbf{x}$ such that $K(\mathbf{x}, t) > K(\mathbf{x}_b, t)$, thereby contradicting our definition of $\mathbf{x}_b$ as the blowup point. From this, it is shown that the time integrals of the viscosity, convective, and ∇p terms of the Navier-Stokes equation must each approach a finite value as $t \to t_b$. This implies that $K(\mathbf{x}_b, t)$ must also approach a finite value as $t \to t_b$, which contradicts our conjecture that $K(\mathbf{x}_b, t) \to \infty$ as $t \to t_b$. Therefore, we have established existence and smoothness of these solutions. Also, we show that the solution for a given set of initial conditions is unique.

1. INTRODUCTION

The Navier-Stokes equation is one of several equations which governs fluid motion. Essentially, it is a statement of Newton's Second Law ($\mathbf{F} = m\mathbf{a}$) applied to the infinitesimal fluid elements, taking into account the pressure gradients and forces due to viscosity. Proving existence and uniqueness of solutions to this equation with various initial conditions and driving forces has been of great interest to the mathematics community (Ref. 1, 2).

In studying the Navier-Stokes equation, many mathematicians have, over the years, developed the concept of "weak" solutions to help gain insight into the behavior of the equation without the need of finding more "exact" solutions, which

may not be possible (Ref. 3). These weak solutions are obtained by relaxing some requirements of the original equation such that solutions are more tractable and easily described. In some cases, it may be possible to demonstrate the existence of a "strong" (or smooth) solution by successive refinements of the weak solutions, or even show that the weak solutions themselves are actually smooth.

In 1934, the French mathematician Jean Leray defined an important class of weak solutions to the Navier-Stokes equation. Instead of working with exact vectors at each point $\mathbf{x} \in \mathbb{R}^3$, the Leray solutions use vector averages over small neighborhoods. Leray showed in his paper that such solutions always exist and never blowup. This achievement opened a new approach to the Navier-Stokes problem. Start with Leray solutions, which you know always exist, and see if you can use them to obtain smooth solutions, which you want to prove always exist.

This and similar methods seem to be the general approach in recent decades to resolving issues about the Navier-Stokes equation and its solutions, including the Millennium Problem sponsored by the Clay Mathematics Institute. Papers implementing these methods, however, are generally extremely difficult to follow, and may be completely understandable only to the authors themselves. This, of course, seriously impedes a proper review of their works since so few are capable of doing so. Also, potential issues in using a particular class of weak solutions may not arise until well after proposed proofs based on these weak solutions have been posted or even published. For example, in October 2018, Tristan Buckmaster and Vlad Vicol of Princeton University showed that under some circumstances, uniqueness of the Leray solutions may break down for the Navier-Stokes equation (Ref. 3, 4). Also, Terance Tao of UCLA constructed a smooth solution to the averaged Navier-Stokes equation that blows up in finite time (Ref. 5).

In this paper, we prove existence and smoothness of solutions to the zero driving-force Navier-Stokes equation for incompressible fluid flow, given a smooth initial fluid velocity profile. We do not, however, follow the general approach of establishing weak solutions, and then somehow showing how to obtain actual or smooth solutions from them (Ref. 6-10). Instead, we conjecture that a smooth solution exists initially but blows up at a point $\mathbf{x}_b$ and time t_b. As the blowup proceeds, a global maximum in the fluid velocity (or equivalently the kinetic energy density $K = \mathbf{u} \cdot \mathbf{u}/2$) must first form at the blowup point. However, as we will see, this global maximum must be finite for all $t > 0$, and therefore our conjectured blowup solution in finite time cannot exist.

Examining the behavior of a solution at its global maximum points seems to be a concept not considered in the current literature. It has the potential advantage over the more conventional approaches for merely establishing existence and smoothness of solutions since it is the global maximum values that determines whether a bounded solution over space and time exists. A complete solution is not necessary. Methods involving weak or approximate solutions over the entire $\mathbb{R}^3$ tend to be highly complicated and difficult to analyze for any meaningful results. This is probably why Dr. Terence Tao claims that the Navier-Stokes global regularity issue is such an "impossible problem" (Ref. 11). For example, the "infamous" nonlinear convective term, which causes so much grief in conventional approaches, vanishes at the global maximum points and therefore isn't an issue with establishing global maximum values.

2. Problem Description and Main Theorem

Written in vector form, the Navier-Stokes equation is given by

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] = \sigma\nabla^2\mathbf{u} - \nabla P + \mathbf{F}(\mathbf{x}, t) \qquad (2.1)$$

where $\mathbf{u}$ is the fluid velocity, ρ is the fluid density, P is pressure, σ is the viscosity coefficient, and $\mathbf{F}$ is the external force per unit volume acting on the fluid elements. In addition to satisfying equation (2.1), a solution $\mathbf{u}$ must also satisfy the equation of continuity, or mass balance, which is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0 \qquad (2.2)$$

This equation states that whatever net fluid mass (per unit time) flows into a fluid element must appear as increased mass of the element, or equivalently, the mass density at that point in the fluid space.

In the problem we are considering, we assume an incompressible fluid, and therefore the density is constant. In this case, we can write equation (2.1) as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu\nabla^2\mathbf{u} - \nabla p + \mathbf{f}(\mathbf{x}, t) \qquad (2.3)$$

where $\nu = \sigma/\rho$ is the normalized viscosity coefficient, $p = (P - P_A)/\rho$ is the normalized pressure, P_A is the ambient pressure (ie. the pressure at infinity), and $\mathbf{f} = \mathbf{F}/\rho$ is the force per unit mass acting on the fluid elements. Also we assume that all external forces acting on the fluid are zero for $t > 0$. That is, we assume that external forces may have acted on the fluid at times $t < 0$, thereby giving rise to an initial fluid velocity profile $\mathbf{u}^0(\mathbf{x})$ at $t = 0$ which we will assume is known. Therefore, equation (2.3) becomes

$$\frac{\partial \mathbf{u}}{\partial t} = \nu\nabla^2\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p \qquad (2.4a)$$

or equivalently

$$\frac{\partial u_i}{\partial t} = \nu\nabla^2 u_i - \sum_{k=1}^{3} u_k\frac{\partial u_i}{\partial x_k} - \frac{\partial p}{\partial x_i} \qquad (2.4b)$$

for our current problem. The initial condition on $\mathbf{u}$ is given by

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}) \quad \text{or} \quad u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}), \ i = 1, 2, 3 \qquad (2.5)$$

where $\mathbf{u}^0(\mathbf{x})$ is a specified vector function of the spatial coordinates. Furthermore, we will assume that $\mathbf{u}^0(\mathbf{x}) \in \mathbb{C}^\infty$ (ie. has continuous partial derivatives to all orders with respect to each spatial variable). For a smooth, physically acceptable solution, we must also assume there exist constants a, C_m, and κ such that

$$\left|\partial_{\mathbf{x}}^m u_i^0(\mathbf{x})\right| \le \max\left|\frac{\partial^{m=m_1+m_2+m_3} u_i^0}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}}\right| \le \frac{a^\kappa C_{mi}^0}{(|\mathbf{x}| + a)^\kappa} \qquad (2.6)$$

where $m = m_1 + m_2 + m_3$, $\partial_{\mathbf{x}}^m$ denotes any m^{th} order spatial derivative, and κ can be any constant greater than $3/2$. This condition ensures that the initial total energy of fluid motion given by

$$E_0 = \int_{\mathbb{R}^3} \frac{1}{2}\left|\mathbf{u}^0(\mathbf{x}, t)\right|^2 d^3\mathbf{x} \qquad (2.7)$$

is finite. To show this, we insert inequality (2.6) into (2.7) and obtain

$$E_0 = \int_{\mathbb{R}^3} \frac{1}{2} \left| \mathbf{u}^0(\mathbf{x}, t) \right|^2 d^3\mathbf{x} \ \leq \ \frac{1}{2} a^{2\kappa} C_0^2 \sum_{i=1}^{3} \int_{\mathbb{R}^3} \frac{d^3\mathbf{x}}{(|\mathbf{x}| + a)^{2\kappa}}$$

$$= \ 2\pi a^{2\kappa} C_0^2 \sum_{i=1}^{3} \int_0^{\infty} \frac{r^2}{(r+a)^{2\kappa}} \ dr \ = \ 6\pi a^{2\kappa} C_0^2 \int_0^{\infty} \frac{r^2}{(r+a)^{2\kappa}} \ dr \qquad (2.8)$$

$$= \ 6\pi a^{2\kappa} C_0^2 \left(\frac{1}{2\kappa - 3} - \frac{1}{\kappa - 1} + \frac{1}{2\kappa - 1} \right) \ = \ \frac{6\pi a^{2\kappa} C_0^2}{(2\kappa - 3)(\kappa - 1)(2\kappa - 1)}$$

From this equation we see that κ must be greater than $3/2$ for a finite E_0. Also, as will be shown later, the pressure gradient magnitude $|\nabla p|$ approaches zero as $a^2/(|\mathbf{x}| + a)^2$ as $|\mathbf{x}| \to \infty$ for any such value of $\kappa > 3/2$. Inserting this result into the Navier-Stokes equation, we then show that it implies the fluid velocity components u_i will not in general approach zero as $|\mathbf{x}| \to \infty$ any faster than $a^2/(|\mathbf{x}| + a)^2$, even if the initial conditions are consistent with values of $\kappa > 2$. Therefore, the range of values of κ that would be compatible with a solution propagated in time is given by $3/2 < \kappa \leq 2$. [1]

Now let us consider the issue of $\nabla \cdot \mathbf{u}$ and the pressure gradient ∇p. Since ρ is constant, we see from equation (2.2) that we must have

$$\nabla \cdot \mathbf{u}(\mathbf{x}, t) \ = \ \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_k}(\mathbf{x}, t) \ = \ 0 \qquad (2.9)$$

in order to satisfy the equation of continuity. Therefore $\mathbf{u}^0(\mathbf{x})$ in equation (2.5) must be a divergence-free vector function. Taking the divergence of both sides of equation (2.4a), we have

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{u}) + \nabla \cdot [(\mathbf{u} \cdot \nabla)\mathbf{u}] \ = \ \nu \nabla^2(\nabla \cdot \mathbf{u}) - \nabla^2 p \qquad (2.10)$$

Inserting equation (2.9) into (2.10), we obtain

$$\nabla^2 p \ = \ -\nabla \cdot [(\mathbf{u} \cdot \nabla)\mathbf{u}] \qquad (2.11)$$

Carrying out the differentiations indicated on the right hand side of equation (2.11), and using equation (2.9), we have

$$\nabla^2 p \ = \ -\sum_{j=1}^{3} \sum_{k=1}^{3} \left(\frac{\partial u_j}{\partial x_k} \right) \left(\frac{\partial u_k}{\partial x_j} \right) \ = \ -Q(\mathbf{x}, t) \qquad (2.12)$$

(See Ref. 2, p. 35, Ref. 12 Ch. 2, Ref. 13) where we have defined

$$Q(\mathbf{x}, t) \ = \ \sum_{j=1}^{3} \sum_{k=1}^{3} \left(\frac{\partial u_j}{\partial x_k}(\mathbf{x}, t) \frac{\partial u_k}{\partial x_j}(\mathbf{x}, t) \right) \qquad (2.13)$$

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[1] In many claims of having solved the Navier-Stokes Millennium Problem, the authors state that the exponent $\kappa > 0$, consistent with the Official Problem Statement (Ref. 1). After this, they proceed with a highly esoteric analysis that probably few others can follow. Even if the reader is totally uninformed about their theory and methods, however, there is one aspect of their arguments that is quite noticeable. That is, the exponent κ is used only in defining initial conditions, and the actual analysis is completely independent of this exponent. Therefore, if their proof is correct, then it seems it would be possible to use their methods to "prove" existence and smoothness of infinite energy solutions which are not physically possible.

Equation (2.12) governs the pressure needed in order to satisfy equation (2.9). If the partial derivatives of the u_j and u_k on the right-hand side of equation (2.12) are known functions of the spatial coordinates x, we can solve this equation as a form1 of Poisson's equation. From potential theory (Ref. 14, 15, 16), the solution is

$$p(\mathbf{x}, t) \;=\; -\int_{\mathbb{R}^3}^* G(\mathbf{x}, \mathbf{x}')\, Q(\mathbf{x}', t)\, d^3\mathbf{x}' \;=\; \frac{1}{4\pi}\int_{\mathbb{R}^3}^* \frac{Q(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|}\, d^3\mathbf{x}' \qquad (2.14)$$

where

$$G(\mathbf{x}, \mathbf{x}') \;=\; -\frac{1}{4\pi}\frac{1}{|\mathbf{x} - \mathbf{x}'|}$$

is the Greens function associated with the Poisson equation and the boundary condition that the solution approach zero as $|\mathbf{x}|$ approaches infinity. The "star" integral over $\mathbb{R}^3$ is equation (2.14) is defined as the following improper integral

$$\int_{\mathbb{R}^3}^* f(\mathbf{x}, \mathbf{x}', t)\, d^3\mathbf{x}' \;=\; \lim_{\epsilon \to 0}\int_{\mathbb{R}^3 - \sigma(\epsilon, \mathbf{x})} f(\mathbf{x}, \mathbf{x}', t)\, d^3\mathbf{x}' \qquad (2.15)$$

where $\sigma(\epsilon, \mathbf{x})$ is defined as a small spherical volume with radius ϵ centered at the point $\mathbf{x}$. That is, $\sigma(\epsilon, \mathbf{x})$ is the set of all points $\mathbf{x}'$ such that $|\mathbf{x} - \mathbf{x}'| < \epsilon$. The region of integration $\mathbb{R}^3 - \sigma(\epsilon, \mathbf{x})$ is then defined as $\mathbb{R}^3$ with the small spherical region $\sigma(\epsilon, \mathbf{x})$ removed. The function $f(\mathbf{x}, \mathbf{x}', t)$ is assumed to be integrable over all $\mathbf{x}' \in \mathbb{R}^3$ except possibly for a singularity at $\mathbf{x}' = \mathbf{x}$. Henceforth in this article, all integrals over $\mathbb{R}^3$ of a function with a singularity at $\mathbf{x} = \mathbf{x}'$ are defined as shown in equation (2.15).

Taking the gradient of both sides of equation (2.14), we have

$$\nabla p(\mathbf{x}, t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}^* Q(\mathbf{x}', t)\, \frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\, d^3\mathbf{x}' \qquad (2.16)$$

Equations (2.14)-(2.16) are used in the next section to establish the existence and spatial profiles of the scalar pressure p and its gradient ∇p, given the spatial profiles of the fluid velocity $\mathbf{u}$.

At this point, we summarize the problem description by stating our main theorem.

If the Cauchy initial value problem for the incompressible Navier-Stokes equation (with normalized viscosity constant $\nu > 0$) is defined as

$$\frac{\partial \mathbf{u}}{\partial t} \;=\; \nu \nabla^2 \mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p \qquad where \qquad \nabla \cdot \mathbf{u}(\mathbf{x}, t) \;=\; 0 \qquad (2.4a)(2.9)$$

with the initial conditions given by

$$\mathbf{u}(\mathbf{x}, 0) \;=\; \mathbf{u}^0(\mathbf{x}) \qquad or \qquad u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}),\ i = 1, 2, 3 \qquad (2.5)$$

where $\mathbf{u}^0(\mathbf{x}) \in \mathbb{C}^\infty$ is a known divergence-free vector field in $\mathbb{R}^3$, and is spatially smooth in the sense that there exists positive constants a, C_{mi}^0, and κ such that

$$\left|\partial_{\mathbf{x}}^m u_i^0(\mathbf{x})\right| \;\leq\; \frac{a^\kappa C_{mi}^0}{(|\mathbf{x}| + a)^\kappa} \qquad m = 0, 1, 2, 3, ... \qquad and \qquad \frac{3}{2} < \kappa \leq 2 \qquad (2.6)$$

then a solution $\mathbf{u}(\mathbf{x}, t), p(\mathbf{x}, t)$ to this problem exists which is also smooth in the sense of the above equation. That is, there exists functions $C_{mi}(t)$ such that

$$\left|\partial_{\mathbf{x}}^m u_i(\mathbf{x}, t)\right| \;\leq\; \frac{a^\kappa C_{mi}(t)}{(|\mathbf{x}| + a)^\kappa} \qquad m = 0, 1, 2, 3, ...$$

These functions need not be continuous, but must be finite for all finite values of t. Furthermore, the solution $\mathbf{u}(\mathbf{x}, t), p(\mathbf{x}, t)$ *is unique.*

Proving this theorem is the objective of this work.

3. EXISTENCE AND UNIQUENESS OF SOLUTION

3.1 Existence and Spatial Dependence of Scalar Pressure Function. Before demonstrating a solution to the Navier-Stokes equation (2.4) with the given initial condition and incompressibility constraint, we must first verify that the scalar pressure function p does in fact exist and has the proper spatial dependence for fluid velocity fields $\mathbf{u}(\mathbf{x}, t)$ that satisfy

$$|u_i(\mathbf{x}, t)| \leq \frac{a^\kappa}{(|\mathbf{x}| + a)^\kappa} A_i[0, 0, 0; 0](t) \tag{3.1}$$

$$\left| \frac{\partial u_i}{\partial x_j}(\mathbf{x}, t) \right| \leq \frac{a^\kappa}{(|\mathbf{x}| + a)^\kappa} A_i[0, 0, 0; j](t) \tag{3.2}$$

and in general

$$\left| \frac{\partial^{m_1+m_2+m_3} u_i}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}}(\mathbf{x}, t) \right| \leq \frac{a^\kappa}{(|\mathbf{x}|) + a)^\kappa} A_i[m_1, m_2, m_3; 0](t) \tag{3.3}$$

$$\left| \frac{\partial}{\partial x_j} \left(\frac{\partial^{m_1+m_2+m_3} u_i}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}}(\mathbf{x}, t) \right) \right| \leq \frac{a^\kappa}{(|\mathbf{x}|) + a)^\kappa} A_i[m_1, m_2, m_3; j](t), \tag{3.4}$$

at some particular time t, where the $A_i[m_1, m_2, m_3; j](t)$ functions may vary with time but not the spatial coordinates. Note that the C_m coefficients from inequality (2.6) can be used as initial values for the $A[m_1, m_2, m_3; j](t)$ functions in (3.1)-(3.4). In this section, these inequalities are taken as a given, and we show that the scalar pressure function p and its gradient ∇p exists for fluid velocity spatial profiles satisfying these boundary conditions "at infinity". In the following sections, we use the initial conditions along with the results of this section to show that solutions $\mathbf{u}(\mathbf{x}, t)$ to the Navier-Stokes equation do in fact satisfy (3.1)-(3.4) for all values of t for which $\mathbf{u}(\mathbf{x}, t)$ remains defined.

We start by obtaining expressions, based on the Poisson integral, for p and its spatial derivatives. Let us choose three non-negative integers m_1, m_2, and m_3, and differentiate equation (2.12) m_1 times with respect to x_1, m_2 times with respect to x_2, and m_3 times with respect to x_3. The result is

$$\nabla^2 \left(\frac{\partial^{m_1+m_2+m_3} p}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}}(\mathbf{x}, t) \right) = - \frac{\partial^{m_1+m_2+m_3} Q}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}}(\mathbf{x}, t) \tag{3.5}$$

Then, using the same potential theory that was used in equation (2.14), we obtain

$$\frac{\partial^{m_1+m_2+m_3} p(\mathbf{x}, t)}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}} = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \frac{\partial^{m_1+m_2+m_3} Q(\mathbf{x}', t)}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}} \, d^3\mathbf{x}' \tag{3.6}$$

We now differentiate equation (2.13) m_1 times with respect to x_1, m_2 times with respect to x_2, and m_3 times with respect to x_3 to obtain

$$\frac{\partial^{m_1+m_2+m_3} Q(\mathbf{x},t)}{\partial x_1{}^{m_1} \partial x_2{}^{m_2} \partial x_3{}^{m_3}} = \sum_{j=0}^{3} \sum_{k=0}^{3} \sum_{\alpha=0}^{m_1} \sum_{\beta=0}^{m_2} \sum_{\gamma=0}^{m_3} \binom{m_1}{\alpha} \binom{m_2}{\beta} \binom{m_3}{\gamma} \qquad (3.7)$$

$$\times \frac{\partial^{m_1+m_2+m_3-\alpha-\beta-\gamma+1} u_j(\mathbf{x},t)}{\partial x_1{}^{m_1-\alpha} \partial x_2{}^{m_2-\beta} \partial x_3{}^{m_3-\gamma} \partial x_k} \frac{\partial^{\alpha+\beta+\gamma+1} u_k(\mathbf{x},t)}{\partial x_1{}^{\alpha} \partial x_2{}^{\beta} \partial x_3{}^{\gamma} \partial x_j}$$

where we have used the Leibnitz rule for determining higher derivatives of the product of two functions. The quantities in parentheses to the right of the summation signs are binomial coefficients. Since, by hypothesis, each of the derivatives on the right-hand side of equation (3.7) approaches zero as $1/(|\mathbf{x}|+a)^\kappa$ as $|\mathbf{x}|$ increases, this equation implies

$$\left| \frac{\partial^{m_1+m_2+m_3} Q(\mathbf{x},t)}{\partial x_1{}^{m_1} \partial x_2{}^{m_2} \partial x_3{}^{m_3}} \right| \leq \frac{a^{2\kappa}}{(|\mathbf{x}|+a)^{2\kappa}} B[m_1,m_2,m_3](t) = \frac{a^{2\kappa}}{(|\mathbf{x}|+a)^{2\kappa}} B(t) \quad (3.8)$$

where

$$B(t) = B[m_1,m_2,m_3](t) = \sum_{j=i}^{3} \sum_{k=1}^{3} \sum_{\alpha=0}^{m_1} \sum_{\beta=0}^{m_2} \sum_{\gamma=0}^{m_3} \binom{m_1}{\alpha} \binom{m_2}{\beta} \binom{m_3}{\gamma}$$

$$\times A_j[m_1-\alpha, m_2-\beta, m_3-\gamma, k](t) \, A_k[\alpha,\beta,\gamma;j](t) \qquad (3.9)$$

and the $A[m_1,m_2,m_3;j](t)$ functions are defined in inequalities (3.3) and (3.4). Taking the absolute value of both sides of equation (3.6) and using the triangle inequality, we have

$$\left| \frac{\partial^{m_1+m_2+m_3} p(\mathbf{x},t)}{\partial^{m_1} x_1 \partial^{m_2} x_2 \partial^{m_3} x_3} \right| \leq \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}-\mathbf{x}'|} \left| \frac{\partial^{m_1+m_2+m_3} Q}{\partial x_1{}^{m_1} \partial x_2{}^{m_2} \partial x_3{}^{m_3}} (\mathbf{x}',t) \right| d^3\mathbf{x}'$$

$$\leq \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}-\mathbf{x}'|} \frac{a^{2\kappa} B(t)}{(|\mathbf{x}'|+a)^{2\kappa}} d^3\mathbf{x}' \qquad (3.10)$$

Expressing the integral on the right-hand side of this inequality in spherical coordinates, we write

$$\left| \frac{\partial^{m_1+m_2+m_3} p(\mathbf{x},t)}{\partial^{m_1} x_1 \partial^{m_2} x_2 \partial^{m_3} x_3} \right| \leq \frac{1}{4\pi} B(t) \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{a^{2\kappa}}{(r'+a)^{2\kappa}} \frac{r'^2 \sin\theta' \, d\phi' d\theta' dr'}{[r^2 + r'^2 - 2rr'\cos\theta']^{1/2}}$$

$$(3.11)$$

Performing the integration over ϕ and making the change of variable $v' = \cos\theta'$ gives us

$$\left| \frac{\partial^{m_1+m_2+m_3} p(\mathbf{x},t)}{\partial^{m_1} x_1 \partial^{m_2} x_2 \partial^{m_3} x_3} \right| \leq \frac{1}{2} B(t) \int_0^\infty \int_{-1}^1 \frac{a^{2\kappa}}{(r'+a)^{2\kappa}} \frac{r'^2}{[r^2+r'^2-2rr'v']^{1/2}} dv' dr' \quad (3.12)$$

We now carry out the integration over v' to obtain

$$\left| \frac{\partial^{m_1+m_2+m_3} p(\mathbf{x},t)}{\partial^{m_1} x_1 \partial^{m_2} x_2 \partial^{m_3} x_3} \right| \leq \frac{1}{2} B(t) \int_0^\infty \frac{a^{2\kappa}}{(r'+a)^{2\kappa}} \frac{[r^2+r'^2-2rr'v']^{1/2} r'^2}{-rr'} \Big|_{-1}^1 dr'$$

$$= \frac{1}{2} B(t) \int_0^\infty \frac{a^{2\kappa}}{(r'+a)^{2\kappa}} \frac{r+r'-|r-r'|}{r} r' dr' \qquad (3.13)$$

$$= \frac{B(t)}{r} \int_0^r \frac{a^{2\kappa}}{(r'+a)^{2\kappa}} r'^2 dr' + B(t) \int_r^\infty \frac{a^{2\kappa}}{(r'+a)^{2\kappa}} \frac{r'^2}{r'} dr'$$

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Since $r' > r$ in the second term on the right-hand side of this inequality, we have

$$
\begin{aligned}
|h| &\leq \frac{B(t)}{r} \int_0^r \frac{a^{2\kappa}}{(r'+a)^{2\kappa}} \, r'^2 \, dr' \;+\; \frac{B(t)}{r} \int_r^\infty \frac{a^{2\kappa}}{(r'+a)^{2\kappa}} \, r'^2 \, dr' \\
&= \frac{B(t)}{r} \int_0^\infty \frac{a^{2\kappa}}{(r'+a)^{2\kappa}} \, r'^2 \, dr' \;=\; \frac{a^3 \, B(t)}{(2\kappa-3)(\kappa-1)(2\kappa-1)\,r}
\end{aligned} \tag{3.14}
$$

From this inequality, we see that p and its spatial derivatives approach zero at least as fast as $1/r$ as r gets larger.

Let us now show that the spatial derivatives of ∇p must approach zero as $1/r^2$ as $r \to \infty$. Differentiating equation (3.4) with respect to x_i $(i = 1, 2, 3)$, we have

$$
\begin{aligned}
\frac{\partial h}{\partial x_i} &= \frac{\partial}{\partial x_i} \left(\frac{\partial^{m_1+m_2+m_3} p}{\partial^{m_1} x_1 \partial^{m_2} x_2 \partial^{m_3} x_3} \right) = \frac{\partial^{m_1+m_2+m_3}}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}} \left(\frac{\partial p}{\partial x_i} \right) \\
&= -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x_i - x_i'}{|\mathbf{x}-\mathbf{x}'|^3} \frac{\partial^{m_1+m_2+m_3} Q}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}} (\mathbf{x}', t) \, d^3\mathbf{x}'
\end{aligned} \tag{3.15}
$$

where we have defined the function h as

$$
h(\mathbf{x},t) = h[m_1, m_2, m_3](\mathbf{x},t) = \frac{\partial^{m_1+m_2+m_3} p}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}}(\mathbf{x},t) \tag{3.16}
$$

Thus far, we have not made any assumptions about the orientation of the coordinate axises. Therefore, let us define our coordinate axises such that the point $\mathbf{x}$ is on the positive x_3 axis. In this case, the radial direction is along $+x_3$, and we may write

$$
\mathbf{x} = r\,\mathbf{e}_3 = r\,\mathbf{e}_r \quad \text{or equivalently} \quad x_1 = 0, \; x_2 = 0, \; x_3 = r \qquad \text{(3.17)-(3.19)}
$$

where $\mathbf{e}_3$ and $\mathbf{e}_r$ are unit vectors in the x_3 and radial directions respectively. For the primed coordinates, we have

$$
x_1' = r' \sin\theta' \cos\phi', \; x_2' = r' \sin\theta' \sin\phi', \; x_3' = r' \cos\theta' \qquad \text{(3.20)-(3.22)}
$$

Inserting equations (3.20)-(3.22) into (3.15) and setting $i = 3$, we obtain

$$
\begin{aligned}
\frac{\partial h}{\partial x_3}(\mathbf{x},t) = &-\frac{1}{4\pi} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{\partial^{m_1+m_2+m_3} Q}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}} (r'\sin\theta'\cos\phi', r'\sin\theta'\sin\phi', r'\cos\theta', t) \\
&\times \frac{(r - r'\cos\theta')\,r'^2 \sin\theta'}{\left[r^2 + r'^2 - 2rr'\cos\theta' \right]^{3/2}} \, d\phi' d\theta' dr' \;=\; \frac{\partial h}{\partial r}
\end{aligned} \tag{3.23}
$$

where we have used equations (3.20)-(3.22) to express the (Cartesian) components of $\mathbf{x}'$ in terms of the primed spherical coordinates. We will later show that this radial component of ∇h is in fact the dominant component in the limit of large values of $|\mathbf{x}|$. Taking the absolute value of both sides of equation (3.23) and using the triangle inequality gives us

$$
\begin{aligned}
\left| \frac{\partial h}{\partial r} \right| \leq &\frac{1}{4\pi} \int_0^\infty \int_0^\pi \int_0^{2\pi} \left| \frac{\partial^{m_1+m_2+m_3} Q}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}} (r'\sin\theta'\cos\phi', r'\sin\theta'\sin\phi', r'\cos\theta', t) \right| \\
&\times \left| \frac{(r - r'\cos\theta')\,r'^2 \sin\theta'}{\left[r^2 + r'^2 - 2rr'\cos\theta' \right]^{3/2}} \right| \, d\phi' d\theta' dr'
\end{aligned} \tag{3.24}
$$

Inserting inequality (3.8) into (3.24), we then have

$$
\left| \frac{\partial h}{\partial r}(\mathbf{x},t) \right| \leq \frac{1}{4\pi} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{a^{2\kappa} B(t)}{(r'+a)^{2\kappa}} \frac{|r - r'\cos\theta'|\, r'^2 \sin\theta'}{\left[r^2 + r'^2 - 2rr'\cos\theta' \right]^{3/2}} \, d\phi' d\theta' dr' \tag{3.25}
$$

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Performing the integration with respect to ϕ' in this inequality, we obtain

$$\left| \frac{\partial h}{\partial r}(\mathbf{x}, t) \right| \leq \frac{1}{2} B(t) \int_0^\infty \int_0^\pi \frac{a^{2\kappa}}{(r'+a)^{2\kappa}} \frac{|r - r' \cos\theta'| \, r'^2 \sin\theta'}{\left[r^2 + r'^2 - rr' \cos\theta' \right]^{3/2}} \, d\theta' \, dr' \qquad (3.26)$$

If we define

$$L(r) = \frac{1}{2} \int_0^\infty \frac{a^{2\kappa}}{(r'+a)^{2\kappa}} \, J(r, r') dr' \qquad (3.27)$$

where

$$J(r, r') = \int_0^\pi \frac{|r - r' \cos\theta'| \, r'^2 \sin\theta'}{\left[r^2 + r'^2 - 2rr' \cos\theta' \right]^{3/2}} d\theta' \qquad (3.28)$$

Then we may write inequality (3.26) as

$$\left| \frac{\partial h}{\partial r}(\mathbf{x}, t) \right| \leq B(t) L(r) = \frac{1}{2} B(t) \int_0^\infty \frac{a^{2\kappa}}{(r'+a)^{2\kappa}} \, J(r, r') \, dr' \qquad (3.29)$$

Let us now evaluate the integral in this equation. We first consider the case of $r' < r$. In this case equation (3.28) can be written as

$$J(r, r') = \int_{-1}^1 \frac{r'^2(r - r'v')}{\left[r^2 + r'^2 - 2rr'v' \right]^{3/2}} \, dv' = \frac{r'^2}{2r} \int_{-1}^1 \frac{2r^2 - 2rr'v'}{\left[r^2 + r'^2 - 2rr'v' \right]^{3/2}} \, dv' \qquad (3.30)$$

$$= \frac{r'^2}{2r} \left[\int_{-1}^1 \frac{r^2 + r'^2 - 2rr'v'}{\left[r^2 + r'^2 - 2rr'v' \right]^{3/2}} \, dv' + \int_{-1}^1 \frac{r^2 - r'^2}{\left[r^2 + r'^2 - 2rr'v' \right]^{3/2}} \, dv' \right]$$

$$= \frac{r'^2}{2r} \left[\int_{-1}^1 \left[r^2 + r'^2 - 2rr'v' \right]^{-1/2} \, dv' + \int_{-1}^1 \frac{r^2 - r'^2}{\left[r^2 + r'^2 - 2rr'v' \right]^{3/2}} \, dv' \right]$$

$$= \frac{r'^2}{2r} \left[\left(-\frac{\left[r^2 + r'^2 - 2rr'v' \right]^{1/2}}{rr'} \right) \Big|_{-1}^1 + \left(\frac{r^2 - r'^2}{rr'} \left[r^2 + r'^2 - 2rr'v' \right]^{-1/2} \right) \Big|_{-1}^1 \right]$$

$$= \frac{r'^2}{2r} \left[2 \left(\frac{r + r' - (r - r')}{rr'} \right) \right] = 2 \frac{r'^2}{r^2}$$

where we have made the change of variable $v' = \cos\theta'$. For $r < r'$, the factor $r - r' \cos\theta'$, whose absolute value appears in equations (3.26)-(3.29), is less than zero for values of $v' = \cos\theta' > r/r'$. Therefore, we must change the sign of the integrand at $v' = r/r'$ when evaluating $J(r, r')$. This function for $r < r'$ then becomes

$$J(r, r') = \int_{-1}^{r/r'} \frac{r'^2(r - r'v')}{\left[r^2 + r'^2 - 2rr'v' \right]^{3/2}} \, dv' - \int_{r/r'}^1 \frac{r'^2(r - r'v')}{\left[r^2 + r'^2 - 2rr'v' \right]^{3/2}} \, dv' \qquad (3.31)$$

$$= \frac{r'^2}{2r} \left[\left(-\frac{\left[r^2 + r'^2 - 2rr'v' \right]^{1/2}}{rr'} \right) \Big|_{-1}^{r/r'} + \left(\frac{r^2 - r'^2}{rr'} \left[r^2 + r'^2 - 2rr'v' \right]^{-1/2} \right) \Big|_{-1}^{r/r'} \right]$$

$$- \frac{r'^2}{2r} \left[- \left(\frac{\left[r^2 + r'^2 - 2rr'v' \right]^{1/2}}{rr'} \right) \Big|_{r/r'}^1 + \left(\frac{r^2 - r'^2}{rr'} \left[r^2 + r'^2 - 2rr'v' \right]^{-1/2} \right) \Big|_{r/r'}^1 \right]$$

$$= 2 \frac{r'^2}{r^2} \left(1 - \sqrt{1 - \left(\frac{r}{r'} \right)^2} \right)$$

Let us check continuity of this function near $r = 0$ by evaluating

$$\lim_{r \to 0} J(r, r') = \lim_{r \to 0} 2 \frac{r'^2}{r^2} \left(1 - \sqrt{1 - \left(\frac{r}{r'} \right)^2} \right) = \lim_{s \to 1} 2 \frac{1 - s}{1 - s^2} = \lim_{s \to 1} \frac{2}{1 + s} = 1 \qquad (3.32)$$

where we have made the change of variable

$$s = \sqrt{1 - \left(\frac{r}{r'}\right)^2}$$

Since $J(r, r')$ has a finite limit as r approaches zero for any value of $r' > r$, this function is continuous and therefore can be integrated with respect to r near $r = 0$. From equations (3.30) and (3.31), we see that

$$J(r, r') \leq 2\frac{r'^2}{r^2} \tag{3.33}$$

if either $r < r'$ or $r > r'$. Inserting inequality (3.33) into equation (3.27) we obtain

$$L(r) \;\leq\; \frac{1}{r^2} \int_0^\infty \frac{a^{2\kappa} r'^2}{(r' + a)^{2\kappa}} \, dr' \;=\; \frac{a^{2\kappa}}{(2\kappa - 3)(\kappa - 1)(2\kappa - 1) \, r^2} \tag{3.34}$$

which shows the $1/r^2$ asymptotic behavior of $L(r)$ in the limit as $r \to \infty$. At first sight of inequality (3.34), one might believe that it implies a singularity exists at $r = 0$. This "singularity", however, is merely an artifact of our gross over-estimation of $J(r, r')$ near $r = 0$. As we have already shown, $J(r, r')$ remains continuous and integrable near $r = 0$. Inserting this result into inequality (3.29) then gives us

$$\left|\frac{\partial h}{\partial r}(\mathbf{x}, t)\right| \leq B(t)L(r) \leq \frac{a^{2\kappa} \, B(t)}{(2\kappa - 3)(\kappa - 1)(2\kappa - 1) \, r^2} = \frac{a^{2\kappa} \, H(\kappa) \, B(t)}{r^2} \tag{3.35}$$

where we have defined

$$H(\kappa) = \frac{1}{(2\kappa - 3)(\kappa - 1)(2\kappa - 1)}$$

Thus, we see that $|\partial h/\partial r|$ approaches zero as $1/r^2$ as $r \to \infty$, and again the left-hand side of this inequality remains bounded and continuous as $r \to 0$.

From inequality (3.8), we see that the non-homogeneous term on the right-hand side of equation (3.5) approaches zero as $1/r^{2\kappa}$ as r increases. According to inequality (3.14), however, h approaches zero as $1/r$ as $r \to \infty$. Therefore, the non-homogeneous term in equation (3.5) can in general be made arbitrarily small compared with the function h and its derivatives by choosing r sufficiently large. This implies that h must approach a harmonic function (ie. solution of Laplace's equation $\nabla^2 h = 0$) in the limit as $r \to \infty$. Let h_L be the (harmonic) function that describes the asymptotic behavior of h as $r \to \infty$. That is h_L is the function to which h approaches as r increases. Since h_L is a harmonic function that approaches zero as $r \to \infty$, it can be written as

$$h_L(r, \theta, \phi) \;=\; \sum_{l=0}^\infty \sum_{m=-l}^l D_{lm} r^{-(l+1)} Y_{lm}(\theta, \phi) \tag{3.36}$$

where the D_{lm} are constants and the Y_{lm} are the spherical harmonics. Taking the gradient of both sides of this equation, we have

$$\nabla h_L = \sum_{l=0}^\infty \sum_{m=-l}^l D_{lm} r^{-(l+2)} \left[-(l+1)Y_{lm}(\theta, \phi)\, \mathbf{e}_r \;+\; \frac{\partial Y_{lm}}{\partial \theta}(\theta, \phi)\, \mathbf{e}_\theta \;+\; \frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial \phi}(\theta, \phi)\, \mathbf{e}_\phi \right]$$
$$\tag{3.37}$$

Examining equations (3.36) and (3.37), we see that the dominate terms (at large values of r) in h_L and ∇h_L are those with $l = m = 0$. Therefore, the asymptotic behavior of h and ∇h can be expressed as

$$h \to \frac{D_{00}}{\sqrt{4\pi} r} \text{ and } \nabla h \to \frac{D_{00}}{\sqrt{4\pi} r^2}\, \mathbf{e}_r \tag{3.38},(3.39)$$

in the limit as $r \to \infty$, with a properly chosen constant D_{00}.[2] Also, note that equations (3.38) and (3.39) are consistent with inequalities (3.13) and (3.35) respectively for large values of r.

From equation (3.39), we see that in the limit as $r = |\mathbf{x}| \to \infty$, ∇h approaches a vector function with only a radial component. This implies that there must be a value r_1 such that for $r > r_1$, we have

$$|\nabla h \cdot \mathbf{e}_\theta| \;<\; |\nabla h \cdot \mathbf{e}_r|, \text{ and } |\nabla h \cdot \mathbf{e}_\phi| \;<\; |\nabla h \cdot \mathbf{e}_r| \qquad (3.40),(3.41)$$

where $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are unit vectors in the polar and azimuthal directions respectively. Therefore $\nabla h \cdot \mathbf{e}_r$, $\nabla h \cdot \mathbf{e}_\theta$, and $\nabla h \cdot \mathbf{e}_\phi$ are the components of ∇h in the radial, polar, and azimuthal directions respectively. The absolute value of ∇h is given by

$$|\nabla h| \;=\; \sqrt{(\nabla h \cdot \mathbf{e}_r)^2 + (\nabla h \cdot \mathbf{e}_\theta)^2 + (\nabla h \cdot \mathbf{e}_\phi)^2} \qquad (3.42)$$

Inserting (3.40) and (3.41) into (3.42), we have

$$|\nabla h| \;=\; \sqrt{(\nabla h \cdot \mathbf{e}_r)^2 + (\nabla h \cdot \mathbf{e}_r)^2 + (\nabla h \cdot \mathbf{e}_r)^2} \qquad (3.43)$$

$$\leq \; \sqrt{3\,(\nabla h \cdot \mathbf{e}_r)^2} \;=\; \sqrt{3}\,|\nabla h \cdot \mathbf{e}_r| \;=\; \sqrt{3}\,\left|\frac{\partial h}{\partial r}\right|$$

for $r > r_1$. Let us define $r_0 = \max[r_1, a]$. We then have from inequalities (3.35) and (3.43)

$$|\nabla h| \;\leq\; \sqrt{3}\,\left|\frac{\partial h}{\partial r}\right| \;\leq\; \frac{\sqrt{3}\,a^{2\kappa} H(\kappa) B(t)}{r^2} \;=\; \frac{a^2}{(2r)^2}\,4\sqrt{3}\,a^{2(\kappa-1)} H(\kappa) B(t) \qquad (3.44)$$

$$\leq \; \frac{a^2}{(r+a)^2}\,4\sqrt{3}\,a^{2(\kappa-1)} H(\kappa) B(t)$$

for $r > r_0$. If $r < r_0$, we define $|\nabla h|_{\max}(t)$ as the maximum of ∇h over the radial interval $0 \leq r \leq r_0$ at time t. Then we may write

$$|\nabla h|(r + r_0)^2 \;\leq\; 4r_0^2 |\nabla h|_{\max}(t) \qquad (3.45)$$

which implies

$$|\nabla h| \;\leq\; \frac{4r_0^2 |\nabla h|_{\max}(t)}{(r+r_0)^2} \;\leq\; \frac{4r_0^2 |\nabla h|_{\max}(t)}{(r+a)^2} \;=\; \frac{a^2}{(r+a)^2}\,\frac{4r_0^2 |\nabla h|_{\max}(t)}{a^2} \qquad (3.46)$$

for $r < r_0$. Combining our results from inequalities (3.44) and (3.46), we have

$$|\nabla h| \;\leq\; \frac{a^2}{(r+a)^2}\,\mathrm{PG}[m_1, m_2, m_3](t) \;=\; \frac{a^2}{(|\mathbf{x}|+a)^2}\,\mathrm{PG}[m_1, m_2, m_3](t) \qquad (3.47)$$

where we have defined

$$\mathrm{PG}[m_1, m_2, m_3](t) \;=\; 4 \max\left[\sqrt{3}\,a^{2(\kappa-1)} H(\kappa) B(t), \frac{r_0^2 |\nabla h|_{\max}(t)}{a^2}\right] \qquad (3.48)$$

and $B(t)$ is given in equation(3.9). Since $|\partial h/\partial x_i| \leq |\nabla h|$ for $i = 1, 2, 3$, inequality (3.47) implies that

$$\left|\frac{\partial^{m_1+m_2+m_3+1} p(\mathbf{x}, t)}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3} \partial x_i}\right| \;\leq\; \left|\nabla\left(\frac{\partial^{m_1+m_2+m_3} p(\mathbf{x}, t)}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}}\right)\right| \;\leq\; \frac{a^2}{(|\mathbf{x}|+a)^2}\,\mathrm{PG}[m_1, m_2, m_3](t)$$
$$(3.49)$$

---

[2]This result is analogous to the dominance of the monopole term in the far-field (ie. large values of $|\mathbf{x}|$) in an electrostatics problem (See Ref. 16, Ch. 4). In such a problem, h corresponds to the electrostatic potential, ∇h corresponds to the electric field, and the right-hand side of equation (3.4) corresponds to the charge density.

and since $\kappa \leq 2$, we can write

$$\left| \frac{\partial^{m_1+m_2+m_3+1} p(\mathbf{x},t)}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3} \partial x_i} \right| \leq \left| \nabla \left(\frac{\partial^{m_1+m_2+m_3} p(\mathbf{x},t)}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}} \right) \right| \leq \frac{a^\kappa}{(|\mathbf{x}|+a)^\kappa} \, \mathrm{PG}[m_1,m_2,m_3](t)$$
(3.50)

Therefore, the components of ∇p and their spatial derivatives to all order satisfy the required boundary conditions.

3.2 Smoothness of Existing Solution. Before establishing existence of a solution to the given problem, we need to show that any such solution must be smooth as defined in our main theorem. Understand that the claims made in this section apply only to pre-blowup times t. That is, if there exists a finite blowup time t_b, then the results of this section apply only to the half-open interval given by $0 \leq t < t_b$. We start by differentiating the Navier-Stokes equation (2.4b) m_1 times with respect to x_1, m_2 times with respect to x_2, and m_3 times with respect to x_3 to obtain

$$\frac{\partial}{\partial t} \left(\frac{\partial^{m_1+m_2+m_3} u_i(\mathbf{x},t)}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}} \right) = \nu \sum_{k=1}^{3} \frac{\partial^{m_1+m_2+m_3+2} u_i(\mathbf{x},t)}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3} \partial x_k^2} - \frac{\partial^{m_1+m_2+m_3+1} p(\mathbf{x},t)}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3} \partial x_i}$$
(3.51)

$$- \sum_{k=1}^{3} \sum_{\alpha=0}^{m_1} \sum_{\beta=0}^{m_2} \sum_{\gamma=0}^{m_3} \binom{m_1}{\alpha} \binom{m_2}{\beta} \binom{m_3}{\gamma} \frac{\partial^{m_1+m_2+m_3-\alpha-\beta-\gamma} u_k(\mathbf{x},t)}{\partial x_1^{m_1-\alpha} \partial x_2^{m_2-\beta} \partial x_3^{m_3-\gamma}} \frac{\partial^{\alpha+\beta+\gamma+1} u_i(\mathbf{x},t)}{\partial x_1^{\alpha} \partial x_2^{\beta} \partial x_3^{\gamma} \partial x_k}$$

Next, we take the absolute value of both sides of this equation and use the triangle inequality to obtain

$$\left| \frac{\partial}{\partial t} \left(\frac{\partial^{m_1+m_2+m_3} u_i(\mathbf{x},t)}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}} \right) \right| \leq \nu \sum_{k=1}^{3} \left| \frac{\partial^{m_1+m_2+m_3+2} u_i(\mathbf{x},t)}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3} \partial x_k^2} \right| + \left| \frac{\partial^{m_1+m_2+m_3+1} p(\mathbf{x},t)}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3} \partial x_i} \right|$$
(3.52)

$$+ \sum_{k=1}^{3} \sum_{\alpha=0}^{m_1} \sum_{\beta=0}^{m_2} \sum_{\gamma=0}^{m_3} \binom{m_1}{\alpha} \binom{m_2}{\beta} \binom{m_3}{\gamma} \left| \frac{\partial^{m_1+m_2+m_3-\alpha-\beta-\gamma} u_k(\mathbf{x},t)}{\partial x_1^{m_1-\alpha} \partial x_2^{m_2-\beta} \partial x_3^{m_3-\gamma}} \right| \left| \frac{\partial^{\alpha+\beta+\gamma+1} u_i(\mathbf{x},t)}{\partial x_1^{\alpha} \partial x_2^{\beta} \partial x_3^{\gamma} \partial x_k} \right|$$

Inserting inequalities (3.1)-(3.4) and (3.50) into (3.52) then gives us

$$\left| \frac{\partial}{\partial t} \left(\frac{\partial^{m_1+m_2+m_3} u_i(\mathbf{x},t)}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}} \right) \right| \leq \nu \frac{a^\kappa}{(|\mathbf{x}|+a)^\kappa} \, \mathrm{L}_i[m_1,m_2,m_3](t)$$
(3.53)

$$+ \frac{a^\kappa}{(|\mathbf{x}|+a)^\kappa} \, \mathrm{PG}[m_1,m_2,m_3](t) + \frac{a^{2\kappa}}{(|\mathbf{x}|+a)^{2\kappa}} \, \mathrm{TT}_i[m_1,m_2,m_3](t)$$

Where we have defined

$$\mathrm{L}_i[m_1,m_2,m_3](t) = A_i[m_1+2,m_2,m_3;0](t) + A_i[m_1,m_2+2,m_3;0](t)$$
(3.54)
$$+ A_i[m_1,m_2,m_3+2;0](t)$$

and

$$\mathrm{TT}_i[m_1,m_2,m_3](t) = \sum_{k=1}^{3} \sum_{\alpha=0}^{m_1} \sum_{\beta=0}^{m_2} \sum_{\gamma=0}^{m_3} \binom{m_1}{\alpha} \binom{m_2}{\beta} \binom{m_3}{\gamma}$$
(3.55)
$$\times A_k[m_1-\alpha,m_2-\beta,m_3-\gamma,0](t) \, A_i[\alpha,\beta,\gamma,k](t)$$

Now since $a/(|\mathbf{x}|+a) < 1$, we can replace the 2κ exponent in the last term on the right-hand side of inequality (3.53) with κ. This gives us

$$\left| \frac{\partial}{\partial t} \left(\frac{\partial^{m_1+m_2+m_3} u_i(\mathbf{x},t)}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}} \right) \right| \leq \frac{a^\kappa}{(|\mathbf{x}|+a)^\kappa} \, F_i[m_1,m_2,m_3](t)$$
(3.56)

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where we have defined

$$F_i[m_1, m_2, m_3](t) = \nu \mathrm{L}_i[m_1, m_2, m_3] + \mathrm{PG}[m_1, m_2, m_3](t) + \mathrm{TT}_i[m_1, m_2, m_3](t)$$

and since

$$\frac{\partial}{\partial t} \left| \frac{\partial^{m_1+m_2+m_3} u_i(\mathbf{x}, t)}{\partial x_1{}^{m_1} \partial x_2{}^{m_2} \partial x_3{}^{m_3}} \right| \leq \left| \frac{\partial}{\partial t} \left(\frac{\partial^{m_1+m_2+m_3} u_i(\mathbf{x}, t)}{\partial x_1{}^{m_1} \partial x_2{}^{m_2} \partial x_3{}^{m_3}} \right) \right| \qquad (3.57)$$

we have from inequality (3.56)

$$\frac{\partial}{\partial t} \left| \frac{\partial^{m_1+m_2+m_3} u_i(\mathbf{x}, t)}{\partial x_1{}^{m_1} \partial x_2{}^{m_2} \partial x_3{}^{m_3}} \right| \leq \frac{a^\kappa}{(|\mathbf{x}| + a)^\kappa} F_i[m_1, m_2, m_3](t) \qquad (3.58)$$

From this inequality, we see that the absolute values of the $u_i(\mathbf{x}, t)$ and their spatial derivatives must be differentiable with respect to time at all time values t for which the $F_i[m_1, m_2, m_3](t)$ functions remain finite. Therefore, we can integrate both sides of this inequality with respect to time to obtain

$$\left| \frac{\partial^{m_1+m_2+m_3} u_i(\mathbf{x}, t)}{\partial x_1{}^{m_1} \partial x_2{}^{m_2} \partial x_3{}^{m_3}} \right| \leq \frac{a^\kappa}{(|\mathbf{x}| + a)^\kappa} \left[\int_0^t F_i[m_1, m_2, m_3](t') \, dt' + C_{mi}^0 \right] \quad (3.59)$$

where $m = m_1 + m_2 + m_3$ and the C_{mi}^0 are the coefficients from the statement of the main theorem used in defining the initial values for the $u_i(\mathbf{x}, t)$ functions. From this inequality, we see that a smooth solution will exist for the u_i and their spatial derivatives if the F_i functions remain finite for finite values of t. Also, note that

$$\int_0^t F_i[m_1, m_2, m_3](t') \, dt' \leq \max_{m=m_1+m_2+m_3} \int_0^t F_i[m_1, m_2, m_3](t') \, dt' \qquad (3.60)$$

where the right-hand side of this inequality denotes the maximum of the time integrals of the $F_i[m_1, m_2, m_3](t)$ functions such that $m = m_1 + m_2 + m_3$. Inserting inequality (3.60) into (3.59), we obtain

$$\left| \frac{\partial^{m_1+m_2+m_3} u_i(\mathbf{x}, t)}{\partial x_1{}^{m_1} \partial x_2{}^{m_2} \partial x_3{}^{m_3}} \right| \leq \frac{a^\kappa}{(|\mathbf{x}| + a)^\kappa} \left[\max_{m=m_1+m_2+m_3} \int_0^t F_i[m_1, m_2, m_3](t') \, dt' + C_{mi}^0 \right]$$
$$(3.61)$$

Therefore, if we set

$$C_{mi}(t) = \max_{m=m_1+m_2+m_3} \int_0^t F_i[m_1, m_2, m_3](t') \, dt' + C_{mi}^0 \qquad (3.62)$$

then we have

$$\left| \frac{\partial^{m_1+m_2+m_3} u_i(\mathbf{x}, t)}{\partial x_1{}^{m_1} \partial x_2{}^{m_2} \partial x_3{}^{m_3}} \right| \leq \frac{a^\kappa}{(|\mathbf{x}| + a)^\kappa} C_{mi}(t) \qquad (3.63)$$

Since the quantity on the right-hand side of inequality (3.62) depends on the sum only of the m_1, m_2, m_3 indices and not the indices individually, the $C_{mi}(t)$ functions defined in equation (3.63) satisfy the requirements of those in the main theorem.

Thus far, we have shown that a solution $\mathbf{u}(\mathbf{x}, t)$ that is initially smooth (according to the hypothesis of the main theorem) will remain smooth for as long as it exists. Also, we have shown that for all times t for which a smooth fluid velocity $\mathbf{u}$ exists, a smooth scalar pressure p also exists. However, we have *not yet* shown that such a solution exists for all $t > 0$. Until proven otherwise, we must recognize the possibility of a "smooth blowup" occurring where the fluid velocity reaches infinite values at some point in finite time, despite the fact that the solution remains smooth at all points prior to the blowup.

3.3 Existence of Pressure Gradient Integral over Time. Before showing that the fluid velocity $\mathbf{u}$ must remain finite for all $t > 0$, we must first establish that the time integral of the scalar pressure gradient ∇p exists over the half-open time interval $0 \leq t < t_b$ (assume the blowup time t_b exists) where $\mathbf{u}$ remains smooth as defined in inequality (3.63). We start with the original Navier-Stokes equation.

$$\frac{\partial u_i}{\partial t} \;=\; \nu \nabla^2 u_i \;-\; \sum_{k=1}^{3} u_k \frac{\partial u_i}{\partial x_k} \;-\; \frac{\partial p}{\partial x_i} \tag{2.4b}$$

Multiplying both sides of this equation by u_i and summing over i, we obtain

$$\sum_{i=1}^{3} u_i \frac{\partial u_i}{\partial t} \;=\; \nu \sum_{i=1}^{3} u_i \nabla^2 u_i \;-\; \sum_{i=1}^{3}\sum_{k=1}^{3} u_i u_k \frac{\partial u_i}{\partial x_k} \;-\; \sum_{i=1}^{3} u_i \frac{\partial p}{\partial x_i} \tag{3.64}$$

Since

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\,u_i^2\right) \;=\; u_i \frac{\partial u_i}{\partial t}$$

Equation (3.64) can be written as

$$\sum_{i=1}^{3} \frac{\partial}{\partial t}\left(\frac{1}{2}\,u_i^2\right) \;=\; \nu \sum_{i=1}^{3} u_i \nabla^2 u_i \;-\; \sum_{i=1}^{3}\sum_{k=1}^{3} u_k \frac{\partial}{\partial x_k}\left(\frac{1}{2}\,u_i^2\right) \;-\; \sum_{i=1}^{3} u_i \frac{\partial p}{\partial x_i} \tag{3.65}$$

From elementary vector analysis, we have

$$\nabla \cdot (u_i \nabla u_i) \;=\; u_i \nabla \cdot (\nabla u_i) \;+\; \nabla u_i \cdot \nabla u_i \;=\; u_i \nabla^2 u_i \;+\; \nabla u_i \cdot \nabla u_i$$

and therefore

$$u_i \nabla^2 u_i \;=\; \nabla \cdot (u_i \nabla u_i) \;-\; \nabla u_i \cdot \nabla u_i \;=\; \nabla^2\left(\frac{1}{2}u_i^2\right) \;-\; \nabla u_i \cdot \nabla u_i \tag{3.66}$$

Inserting this result into equation (3.65), we obtain

$$\sum_{i=1}^{3} \frac{\partial}{\partial t}\left(\frac{1}{2}\,u_i^2\right) = \nu \sum_{i=1}^{3} \nabla^2\left(\frac{1}{2}\,u_i^2\right) - \nu \sum_{i=1}^{3} \nabla u_i \cdot \nabla u_i - \sum_{i=1}^{3}\sum_{k=1}^{3} u_k \frac{\partial}{\partial x_k}\left(\frac{1}{2}\,u_i^2\right) - \sum_{i=1}^{3} u_i \frac{\partial p}{\partial x_i} \tag{3.67}$$

If we define the energy density of fluid motion K as

$$K(\mathbf{x},t) = \frac{1}{2}\sum_{i=1}^{3}\left(u_i(\mathbf{x},t)\right)^2 = \frac{1}{2}\mathbf{u}(\mathbf{x},t)\cdot\mathbf{u}(\mathbf{x},t) \tag{3.68}$$

equation (3.67) can be written as

$$\frac{\partial K}{\partial t} \;=\; \nu\,\nabla^2 K \;-\; \nu \sum_{i=1}^{3} \nabla u_i \cdot \nabla u_i \;-\; \sum_{i=1}^{3} u_i \frac{\partial K}{\partial x_i} \;-\; \sum_{i=1}^{3} u_i \frac{\partial p}{\partial x_i}$$

or equivalently

$$\frac{\partial K}{\partial t} \;=\; \nu \nabla^2 K - \nu \sum_{i=1}^{3} \nabla u_i \cdot \nabla u_i - \mathbf{u}\cdot\nabla K - \mathbf{u}\cdot\nabla p \tag{3.69}$$

$$= \; \nu \nabla \cdot (\nabla K) - \nu \sum_{i=1}^{3}(\nabla u_i \cdot \nabla u_i) - \nabla \cdot [(K+p)\mathbf{u}]$$

where we have used the fact that $\nabla \cdot \mathbf{u} = 0$ in the last step. Let us now define the total energy of fluid motion as

$$E(t) \;=\; \int_{\mathbb{R}^3} K(\mathbf{x}, t)\, d^3\mathbf{x} \tag{3.70}$$

The initial value E_0 of this function was shown to be finite in equation (2.8). Let us examine the evolution of the function $E(t)$. Integrating equation (3.69) over $\mathbb{R}^3$ and using (3.70) gives us

$$\frac{dE}{dt} \;=\; \nu \int_{\mathbb{R}^3} \nabla \cdot (\nabla K)\, d^3\mathbf{x} \;-\; \nu \sum_{i=1}^{3} \int_{\mathbb{R}^3} (\nabla u_i \cdot \nabla u_i)\, d^3\mathbf{x} \;-\; \int_{\mathbb{R}^3} \nabla \cdot [(p + K)\mathbf{u})]\, d^3\mathbf{x} \tag{3.71}$$

We now show that the first and third terms on the right-hand side of equation (3.71) vanish. Integrating equation (3.69) over the spherical region in $\mathbb{R}^3$ defined by $|\mathbf{x}| \leq R$ we have

$$\frac{\partial}{\partial t} \int_{|\mathbf{x}| \leq R} K(\mathbf{x}, t)\, d^3\mathbf{x} \;=\; \nu \int_{|\mathbf{x}| \leq R} \nabla \cdot (\nabla K)\, d^3\mathbf{x} \;-\; \nu \sum_{i=1}^{3} \int_{|\mathbf{x}| \leq R} (\nabla u_i \cdot \nabla u_i)\, d^3\mathbf{x}$$
$$-\; \int_{|\mathbf{x}| \leq R} \nabla \cdot [(p + K)\mathbf{u}]\, d^3\mathbf{x} \tag{3.72}$$

Applying the divergence theorem to the first and third terms on the right-hand side of equation (3.72), we have

$$\frac{\partial}{\partial t} \int_{|\mathbf{x}| \leq R} K(\mathbf{x}, t)\, d^3\mathbf{x} \;=\; \nu \int_{|\mathbf{x}| = R} \nabla K \cdot \mathbf{e}_r\, dS \;-\; \nu \sum_{i=1}^{3} \int_{|\mathbf{x}| \leq R} (\nabla u_i \cdot \nabla u_i)\, d^3\mathbf{x}$$
$$-\; \int_{|\mathbf{x}| = R} (p + K)\, \mathbf{u} \cdot \mathbf{e}_r\, dS \tag{3.73}$$

where $\mathbf{e}_r$ is the unit vector in the radial direction. Differentiating both sides of equation (3.68) with respect to x_j gives us

$$\frac{\partial K}{\partial x_j}(\mathbf{x}, t) \;=\; \sum_{i=1}^{3} u_i(\mathbf{x}, t)\, \frac{\partial u_i}{\partial x_j}(\mathbf{x}, t) \tag{3.74}$$

Since the function $\mathbf{u}$ must be consistent with inequality (3.63), we take the absolute value of both sides of equation (3.74) and use inequality (3.63) along with the triangle inequality to obtain

$$\left| \frac{\partial K}{\partial x_j}(\mathbf{x}, t) \right| \;\leq\; \sum_{i=1}^{3} |u_i(\mathbf{x}, t)|\, \left| \frac{\partial u_i}{\partial x_j}(\mathbf{x}, t) \right| \tag{3.75}$$
$$\leq\; \frac{a^{2\kappa}}{(|\mathbf{x}| + a)^{2\kappa}} \sum_{i=1}^{3} A_i[0, 0, 0; 0](t)\, A_i[0, 0, 0; j](t)$$

From this inequality, we have

$$|\nabla K(\mathbf{x}, t)| \;\leq\; \sum_{i=1}^{3} \left| \frac{\partial K}{\partial x_i}(\mathbf{x}, t) \right| \;\leq\; \sum_{i=1}^{3} \sum_{j=1}^{3} |u_i(\mathbf{x}, t)|\, \left| \frac{\partial u_i}{\partial x_j}(\mathbf{x}, t) \right| \tag{3.76}$$
$$\leq\; \frac{a^{2\kappa}}{(|\mathbf{x}| + a)^{2\kappa}} \sum_{i=1}^{3} \sum_{j=1}^{3} A_i[0, 0, 0; 0](t)\, A_i[0, 0, 0, j](t)$$

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Applying inequality (3.76) to the first integral on the right-hand side of equation (3.73) gives us

$$\left| \int_{|\mathbf{x}|=R} \nabla K \cdot \mathbf{e}_r \, dS \right| \leq \sum_{i=1}^{3} \sum_{j=1}^{3} A_i[0,0,0;0](t) \, A_i[0,0,0;j](t) \int_{|\mathbf{x}|=R} \frac{a^{2\kappa}}{(|\mathbf{x}|+a)^{2\kappa}} \, dS \tag{3.77}$$

$$= \frac{4\pi R^2 a^{2\kappa}}{(R+a)^{2\kappa}} \sum_{i=1}^{3} \sum_{j=1}^{3} A_i[0,0,0;0](t) \, A_i[0,0,0;j](t)$$

Taking the limit of both sides of this inequality as $R \to \infty$, we obtain

$$\lim_{R \to \infty} \int_{|\mathbf{x}|=R} \nabla K \cdot \mathbf{e}_r \, dS = 0 \tag{3.78}$$

From the last term on the right-hand side of equation (3.73), we have

$$\left| \int_{|\mathbf{x}|=R} (p+K) \, \mathbf{u} \cdot \mathbf{e}_r \, dS \right| \leq \int_{|\mathbf{x}|=R} |p+K||\mathbf{u}| \, dS \tag{3.79}$$

$$\leq \int_{|\mathbf{x}|=R} \left(\overline{|p|}(R) + \overline{K}(R) \right) \frac{a^\kappa \, A\,[|\mathbf{u}|]}{(R+a)^\kappa} \, dS$$

$$= 4\pi R^2 \left(\overline{|p|}(R) + \overline{K}(R) \right) \frac{a^\kappa \, A\,[|\mathbf{u}|]}{(R+a)^\kappa}$$

where we have defined $\overline{|p|}(R)$ and $\overline{K}(R)$ as the average values of $|p|$ and K respectively for $|\mathbf{x}| = R$. Also, we have defined the fluid velocity magnitude coefficient $A[|\mathbf{u}|](t)$ as

$$A[|\mathbf{u}|](t) = \sqrt{\sum_{k=1}^{3} A_k^2[0,0,0;0](t)} \quad \text{so that} \quad |\mathbf{u}(\mathbf{x},t)| \leq \frac{a^\kappa}{(|\mathbf{x}|+a)^\kappa} \, A\,[|\mathbf{u}|]$$

where the $A_k[0,0,0;0](t)$ are from equation (3.1). In the right-hand side of inequality (3.79), the scalar pressure p approaches zero as $1/R$ as $R \to \infty$, and the kinetic energy density K approaches zero as $1/R^{2\kappa}$ as $R \to \infty$. Therefore, the first term of the right-hand side of this inequality approaches zero as $1/R^{\kappa-1}$ as $R \to \infty$, and the second term approaches zero as $1/R^{3\kappa-2}$ as $R \to \infty$. Since $\kappa > 3/2$, it follows then that both terms on the right-hand side of inequality (3.79) vanish as $R \to \infty$. Therefore, we have

$$\lim_{R \to \infty} \int_{|\mathbf{x}|=R} (p+K) \, \mathbf{u} \cdot \mathbf{e}_r \, dS = 0 \tag{3.80}$$

Now we take the limit as $R \to \infty$ of both sides of equation (3.73), and use (3.78) and (3.80) to obtain

$$\frac{dE}{dt} = -\nu \sum_{i=1}^{3} \int_{\mathbb{R}^3} (\nabla u_i \cdot \nabla u_i) \, d^3\mathbf{x} = -\nu \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{\mathbb{R}^3} \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x},t) \right)^2 d^3\mathbf{x} \tag{3.81}$$

Integrating equation (3.81) with respect to time gives us

$$E(t) = E_0 - \nu \sum_{i=1}^{3} \sum_{j=1}^{3} \int_0^t \int_{\mathbb{R}^3} \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x},t') \right)^2 d^3\mathbf{x} \, dt' \leq E_0 \tag{3.82}$$

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Since $E(t) \geq 0$ and $\nu > 0$, the summation of the integrals in equation (3.82) must be finite, and since each these of these integrals is positive, they must all be finite. Therefore, we may write

$$\int_0^t \int_{\mathbb{R}^3} \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \right)^2 \, d^3\mathbf{x} \, dt' \; = \; W_{ij}(t) \tag{3.83}$$

where each of the $W_{ij}(t)$ functions are finite for all t, provided only that the fluid velocity $\mathbf{u}(\mathbf{x}, t)$ remains smooth as defined in inequality (3.63). Let us now establish a connection between the $W_{ij}(t)$ functions and the time integral of the scalar pressure gradient. We first note that since the functions being integrated in equations (3.82) and (3.83) are everywhere greater than or equal to zero, we may write

$$\int_0^t \int_{S^3(t')} \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \right)^2 \, d^3\mathbf{x} \, dt' \; \leq \; \int_0^t \int_{\mathbb{R}^3} \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \right)^2 \, d^3\mathbf{x} \, dt' \; = \; W_{ij}(t) \tag{3.84}$$

where $S^3(t)$ can be any subset of $\mathbb{R}^3$ which may change with time. Let us now show that

$$\int_0^t \int_{\mathbb{R}^3} \left| \frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \, \frac{\partial u_j}{\partial x_i}(\mathbf{x}, t') \right| \, d^3\mathbf{x} \, dt' \; \leq \; W_{ij}(t) + W_{ji}(t) \quad i, j = 1, 2, 3 \tag{3.85}$$

for all t such that $\mathbf{u}(\mathbf{x}, t)$ remains smooth. We first define $S_{ij}^3(t)$ as the subset of $\mathbb{R}^3$ (at time t) where $|\partial u_i / \partial x_j|$ is greater than or equal to $|\partial u_j / \partial x_i|$. We may then write

$$\int_0^t \int_{\mathbb{R}^3} \left| \frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \right| \left| \frac{\partial u_j}{\partial x_i}(\mathbf{x}, t') \right| \, d^3\mathbf{x} \, dt' \; \leq \; \int_0^t \int_{S_{ij}^3(t')} \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \right)^2 \, d^3\mathbf{x} \, dt' \tag{3.86}$$

$$+ \int_0^t \int_{\mathbb{R}^3 - S_{ij}^3(t')} \left(\frac{\partial u_j}{\partial x_i}(\mathbf{x}, t') \right)^2 \, d^3\mathbf{x} \, dt'$$

Since both integrands on the right-hand side of inequality (3.86) are positive and the subsets $S_{ij}^3(t)$ and $\mathbb{R}^3 - S_{ij}^3(t)$ are both contained within $\mathbb{R}^3$ for any time t, we have

$$\int_0^t \int_{S_{ij}^3(t')} \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \right)^2 \, d^3\mathbf{x} \, dt' \; \leq \; \int_0^t \int_{\mathbb{R}^3} \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \right)^2 \, d^3\mathbf{x} \, dt' \tag{3.87}$$

and

$$\int_0^t \int_{\mathbb{R}^3 - S_{ij}^3(t')} \left(\frac{\partial u_j}{\partial x_i}(\mathbf{x}, t') \right)^2 \, d^3\mathbf{x} \, dt' \; \leq \; \int_0^t \int_{\mathbb{R}^3} \left(\frac{\partial u_j}{\partial x_i}(\mathbf{x}, t') \right)^2 \, d^3\mathbf{x} \, dt' \tag{3.88}$$

Inserting these into inequality (3.87) then gives us

$$\int_0^t \int_{\mathbb{R}^3} \left| \frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \, \frac{\partial u_j}{\partial x_i}(\mathbf{x}, t') \right| \, d^3\mathbf{x} \, dt' \; \leq \; \int_0^t \int_{\mathbb{R}^3} \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \right)^2 \, d^3\mathbf{x} \, dt' \tag{3.89}$$

$$+ \int_0^t \int_{\mathbb{R}^3} \left(\frac{\partial u_j}{\partial x_i}(\mathbf{x}, t') \right)^2 \, d^3\mathbf{x} \, dt'$$

From the definition of the $W_{ij}(t)$ functions in equation (3.83), inequality (3.89) can be written as

$$\int_0^t \int_{\mathbb{R}^3} \left| \frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \, \frac{\partial u_j}{\partial x_i}(\mathbf{x}, t') \right| \, d^3\mathbf{x} \, dt' \; \leq \; W_{ij}(t) + W_{ji}(t) \tag{3.85}$$

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thereby proving inequality (3.85). Applying the triangle inequality to equation (2.13), we have

$$|Q(\mathbf{x},t)| \ \leq\ \sum_{i=1}^{3}\sum_{j=1}^{3}\left|\frac{\partial u_i}{\partial x_j}(\mathbf{x},t)\ \frac{\partial u_j}{\partial x_i}(\mathbf{x},t)\right| \tag{3.90}$$

Integrating this inequality over $\mathbb{R}^3$ and t such that $\mathbf{u}(\mathbf{x},t)$ remains smooth, and using inequality (3.85) then gives us

$$
\begin{aligned}
\int_0^t \int_{\mathbb{R}^3} |Q(\mathbf{x},t')|\, d^3\mathbf{x}\, dt' \ &=\ \int_{\mathbb{R}^3}\int_0^t |Q(\mathbf{x},t')|\, dt'\, d^3\mathbf{x} \\
&\leq\ \sum_{i=1}^{3}\sum_{j=1}^{3}\int_0^t \int_{\mathbb{R}^3} \left|\frac{\partial u_i}{\partial x_j}(\mathbf{x},t')\frac{\partial u_j}{\partial x_i}(\mathbf{x},t')\right|\, d^3\mathbf{x}\, dt' \\
&\leq\ \sum_{i=1}^{3}\sum_{j=1}^{3}[W_{ij}(t)\ +\ W_{ji}(t)]\ =\ 2\sum_{i=1}^{3}\sum_{j=1}^{3}W_{ij}(t)
\end{aligned}
\tag{3.91}
$$

where we have reversed the order of integration over space and time. This is valid since the solution $\mathbf{u}(\mathbf{x},t)$ and its spatial derivatives are smooth prior to any blowup. Let us now define the function

$$q(\mathbf{x},t) = \int_0^t |Q(\mathbf{x},t')|\, dt' \tag{3.92}$$

Inequality (3.91) can then be written as

$$\int_{\mathbb{R}^3} q(\mathbf{x},t)\, d^3\mathbf{x} \ \leq\ 2\sum_{i=1}^{3}\sum_{j=1}^{3}W_{ij}(t)\ =\ \frac{2}{\nu}\left[E_0 - E(t)\right]\ \leq\ \frac{2E_0}{\nu} \tag{3.93}$$

where we have also used equations (3.82) and (3.83). From this inequality, we see that the integral of q over $\mathbb{R}^3$, and therefore any subset thereof, is uniformly bounded above for all time t such that the fluid velocity $\mathbf{u}(\mathbf{x},t)$ remains smooth.

At this point, we show that inequality (3.93) implies that the time integral of $|\nabla p|$ must remain finite for all $\mathbf{x} \in \mathbb{R}^3$ regardless of how large $\mathbf{u}(\mathbf{x},t)$ becomes, provided only that it remains smooth according to inequality (3.63). Applying the triangle inequality to equation (2.16), we have

$$|\nabla p(\mathbf{x},t)| \ \leq\ \frac{1}{4\pi}\int_{\mathbb{R}^3}^{*} |Q(\mathbf{x}',t)|\,\frac{|\mathbf{x}-\mathbf{x}'|}{|\mathbf{x}-\mathbf{x}'|^3}\, d^3\mathbf{x}' \ =\ \frac{1}{4\pi}\int_{\mathbb{R}^3}^{*} \frac{|Q(\mathbf{x}',t)|}{|\mathbf{x}-\mathbf{x}'|^2}\, d^3\mathbf{x}' \tag{3.94}$$

where we have defined the above improper integral in the manner shown in equation (2.15). Integrating both sides of this inequality with respect time gives us

$$
\begin{aligned}
\Lambda(\mathbf{x},t) \ =\ \int_0^t |\nabla p(\mathbf{x},t')|\, dt' &\leq\ \frac{1}{4\pi}\int_0^t\int_{\mathbb{R}^3}^{*} \frac{|Q(\mathbf{x}',t')|}{|\mathbf{x}-\mathbf{x}'|^2}\, d^3\mathbf{x}'\, dt' \\
&=\ \frac{1}{4\pi}\int_{\mathbb{R}^3}^{*}\int_0^t \frac{|Q(\mathbf{x}',t')|}{|\mathbf{x}-\mathbf{x}'|^2}\, dt'\, d^3\mathbf{x}' \ =\ \frac{1}{4\pi}\int_{\mathbb{R}^3}^{*} \frac{1}{|\mathbf{x}-\mathbf{x}'|^2}\int_0^t |Q(\mathbf{x}',t')|\, dt'\, d^3\mathbf{x}' \\
&=\ \frac{1}{4\pi}\int_{\mathbb{R}^3}^{*} \frac{q(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|^2}\, d^3\mathbf{x}'
\end{aligned}
\tag{3.95}
$$

where we have again reversed the order of integration over space and time, and have defined

$$\Lambda(\mathbf{x},t) \ =\ \int_0^t |\nabla p(\mathbf{x},t')|\, dt' \tag{3.96}$$

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as the time integral of $|\nabla p(\mathbf{x}, t)|$. Now, the integral on the right-hand side of inequality (3.95) can be written as

$$\int_{\mathbb{R}^3}^* \frac{q(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|^2} \, d^3\mathbf{x}' = \int_{|\mathbf{x}-\mathbf{x}'|>R} \frac{q(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|^2} \, d^3\mathbf{x}' + \int_{|\mathbf{x}-\mathbf{x}'|\leq R}^* \frac{q(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|^2} \, d^3\mathbf{x}' \qquad (3.97)$$

where R can be any number greater than zero, and we have defined the improper integral

$$\int_{|\mathbf{x}-\mathbf{x}'|\leq R}^* \frac{q(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|^2} \, d^3\mathbf{x}' = \lim_{\epsilon \to 0} \int_{\epsilon \leq |\mathbf{x}-\mathbf{x}'| \leq R} \frac{q(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|^2} \, d^3\mathbf{x}'$$

For the first integral on the right-hand side of equation (3.97), we have

$$\int_{|\mathbf{x}-\mathbf{x}'|>R} \frac{q(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|^2} \, d^3\mathbf{x}' \leq \int_{|\mathbf{x}-\mathbf{x}'|>R} \frac{q(\mathbf{x}', t)}{R^2} \, d^3\mathbf{x}' \qquad (3.98)$$

$$= \frac{1}{R^2} \int_{|\mathbf{x}-\mathbf{x}'|>R} q(\mathbf{x}', t) \, d^3\mathbf{x}' \leq \frac{1}{R^2} \int_{\mathbb{R}^3} q(\mathbf{x}', t) \, d^3\mathbf{x}' \leq \frac{2E_0}{\nu R^2} = \text{finite}$$

and therefore this integral is finite for any $t \geq 0$.

For the second integral on the right-hand side of equation (3.97), we first define a set of primed coordinates such that

$$\mathbf{x}' = \mathbf{x} + r' \sin\theta' \cos\phi' \mathbf{e}_1 + r' \sin\theta' \sin\phi' \mathbf{e}_2 + r' \cos\theta' \mathbf{e}_3$$

Then this integral can be written as

$$\int_{|\mathbf{x}-\mathbf{x}'|\leq R}^* \frac{q(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|^2} \, d^3\mathbf{x}' = \lim_{\epsilon \to 0} \int_\epsilon^R \int_0^\pi \int_0^{2\pi} \frac{1}{r'^2} q(\mathbf{x}'(r', \theta', \phi', t)) \, r'^2 \sin\theta' \, d\phi' \, d\theta' \, dr'$$

$$= \lim_{\epsilon \to 0} \int_\epsilon^R \int_0^\pi \int_0^{2\pi} q(\mathbf{x}'(r', \theta', \phi', t)) \sin\theta' \, d\phi' \, d\theta' \, dr' \qquad (3.99)$$

$$= \lim_{\epsilon \to 0} \int_\epsilon^R \int_\Omega q(\mathbf{x}'(r', \theta', \phi', t)) \, d\Omega' \, dr'$$

where the inside integral is over solid angle of a sphere centered at the field point $\mathbf{x}$. Now, from inequality (3.93), we see that $q(\mathbf{x}', t)$ is integrable over all $\mathbf{x}' \in \mathbb{R}^3$ and pre-blowup time t. This is true even if q approaches spatial singularities as t approaches a finite blowup time. Therefore, the integral of $q(\mathbf{x}'(r', \theta', \phi'), t)$ over any finite range of spherical coordinates must also be finite, provided $r' > 0$. [3] This implies that the integral on the right-hand side of equation (3.99) must be finite for all $\epsilon > 0$, and can therefore be used in evaluating the limit of this integral as $\epsilon \to 0$. Furthermore, in taking the limit as $\epsilon \to 0$ in this equation, we are excluding the effect of any singularity occurring in $q(\mathbf{x}', t)$ at the field point $\mathbf{x}$. Therefore, we may write

$$\int_{|\mathbf{x}-\mathbf{x}'|\leq R}^* \frac{q(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|^2} \, d^3\mathbf{x}' = \text{finite} \qquad (3.100)$$

Inserting equations (3.97)-(3.100) into equation (3.95), we then have

$$\Lambda(\mathbf{x}, t) = \int_0^t |\nabla p(\mathbf{x}, t')| \, dt' \leq \frac{1}{4\pi} \int_{|\mathbf{x}-\mathbf{x}'|\leq R}^* \frac{q(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|^2} \, d^3\mathbf{x}' + \frac{E_0}{2\pi\nu R^2} \qquad (3.101)$$

$$= \text{finite}$$

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[3]This is because at $r' = 0$, the transformation of the integral from Cartesian (or rectangular) coordinates to spherical coordinates becomes undefined (ie. the Jacobian factor $r'^2 \sin\theta'$ vanishes).

The fact that $\Lambda(\mathbf{x}, t)$ remains finite as $t \to t_b$ despite a possible blowup in $q(\mathbf{x}, t)$ at this time can also be shown as follows. From equation (3.92), we see that if q blows up at field point $\mathbf{x}$ and time t_b, then Q must do so also. Then, from equation (2.14), we see that a blowup in $Q(\mathbf{x}, t)$ as $t \to t_b$ implies that the scalar pressure p also has a blowup at this point and time.

Now, in the discussion following equation (3.63), we see that a blowup in p must be a "smooth blowup" where p and all of its spatial derivatives of are defined at all times $t < t_b$. In this case, a global maximum in $|p|$ must form at the blowup point $\mathbf{x}$ immediately before the blowup time. Otherwise, the blowup would occur first at some other point $\mathbf{x}^*$ such that $|p(\mathbf{x}^*, t)| > |p(\mathbf{x}, t)|$. But since p is a spatially smooth function up to the blowup time t_b, it's extremum values can only occur where $\nabla p = 0$. Therefore, we must have $|\nabla p(\mathbf{x}, t_b)| = 0$, where $\mathbf{x}$ is the field and blowup point. This implies that $|\nabla p(\mathbf{x}, t)|$ is bounded and continuous on the closed time interval $0 \le t \le t_b$, and therefore the time integral of $|\nabla p(\mathbf{x}, t)|$ over this time interval must be finite.

It should be noted here that although we have shown that $\Lambda(\mathbf{x}, t)$ is defined (ie. finite) for all $\mathbf{x} \in \mathbb{R}^3$ and $t \ge 0$, we have not established an upper bound for Λ. This can be a point of confusion for some readers who tend to believe that it is necessary to show an upper bound to Λ in order to prove it is defined. This, however, is not true as can be seen in the following example where q is taken as proportional to the Dirac δ function. In this case, we have $\Lambda = 1/|\mathbf{x} - \mathbf{x}'|^2$ if $\mathbf{x} \neq \mathbf{x}'$ and $\Lambda = $ (finite) if $\mathbf{x} = \mathbf{x}'$. This function is defined but certainly not bounded for all $\mathbf{x}$ and $\mathbf{x}' \in \mathbb{R}^3$.

For those familiar with electrostatics, this problem is analogous to calculating the electric field $\mathbf{E}(\mathbf{x})$ inside a charge distribution. This field is determined by carrying out a vector sum of the fields due to the charges at *other points* $\mathbf{x}'$ in the distribution. The contribution to $\mathbf{E}(\mathbf{x})$ from any charge at the field point $\mathbf{x}$ is excluded from the summation, even if there is a point charge at $\mathbf{x}$. In this case, $\mathbf{E}$ would be very strong at points near $\mathbf{x}$, but this charge would contribute nothing to the field $\mathbf{E}$ at $\mathbf{x}$. Otherwise, the charge would be exerting a force on itself, causing its own acceleration, which is physically impossible.

At this point, we have shown that $\Lambda(\mathbf{x}, t)$ defined in equation (3.96) exists (ie. is finite) for all $\mathbf{x} \in \mathbb{R}^3$ and $t \ge 0$. This plays a vital role in proving existence and smoothness of the solutions $\mathbf{u}$ and p, as we will see in the next section.

3.4 Existence and Smoothness of Solution over Time. At this point, we show that a solution $\mathbf{u}(\mathbf{x}, t)$ consistent with the initial conditions specified in the problem description will exist and remain smooth for all $t \ge 0$. That is, no blowups occur where the solution becomes infinite at some location in finite time.

We conjecture that a blowup does occur at some point $\mathbf{x}_b \in \mathbb{R}^3$ and finite time t_b, and then show that this conjecture leads to a contradiction. Now, the solution $\mathbf{u}(\mathbf{x}, t)$ must remain smooth at all points $\mathbf{x} \in \mathbb{R}^3$ and all times $t < t_b$ prior to the blowup. Although a blowup may be in progress, the solution $\mathbf{u}$ still exists and must therefore be smooth in the sense of inequality (3.63). This means that the blowup point $\mathbf{x}_b$ must evolve from a global maximum point. Otherwise, the blowup would occur first at some other point $\mathbf{x}$ such that $|\mathbf{u}(\mathbf{x}, t)| > |\mathbf{u}(\mathbf{x}_b, t)|$, thereby contradicting our definition of the blowup point $\mathbf{x}_b$. Therefore, we define t_m as the time a global maximum forms at $\mathbf{x}_b$, leading to the blowup at time t_b. Also, understand that for $t_m \le t < t_b$, the global maximum of K must remain at $\mathbf{x}_b$. Otherwise, the blowup would start at the new maximum point, thereby

contradicting our definition of $\mathbf{x}_b$ as the blowup point. As $t \to t_b$, $\mathbf{u}(\mathbf{x}_b, t)$ would reach arbitrarily high values as a spatially smooth function for t sufficiently close to t_b. This is how a smooth blowup is defined, which is the only type of blowup that is possible in this problem.

Let us now determine if $\mathbf{u}(\mathbf{x}_b, t)$ does reach such high values as $t \to t_b$. Inserting $\mathbf{x}_b$ into equation (3.69) and integrating with respect to time gives us

$$K(\mathbf{x}_b, t) - K(\mathbf{x}_b, 0) = \int_0^t \left[\nu \nabla^2 K(\mathbf{x}_b, t') - \nu \sum_{i=1}^3 |\nabla u_i(\mathbf{x}_b, t')|^2 \right. \tag{3.102}$$
$$\left. - \; \mathbf{u}(\mathbf{x}_b, t') \cdot \nabla K(\mathbf{x}_b, t') - \mathbf{u}(\mathbf{x}_b, t') \cdot \nabla p(\mathbf{x}_b, t') \right] dt'$$

If $t > t_m$, equation (3.102) may be written as

$$K(\mathbf{x}_b, t) \;=\; K(\mathbf{x}_b, t_m) \;+\; K_b(t) \tag{3.103}$$

where we have defined

$$K_b(t) = \int_{t_m}^t \left[\nu \nabla^2 K(\mathbf{x}_b, t') - \nu \sum_{i=1}^3 |\nabla u_i(\mathbf{x}_b, t')|^2 \right. \tag{3.104}$$
$$\left. - \; \mathbf{u}(\mathbf{x}_b, t') \cdot \nabla K(\mathbf{x}_b, t') - \mathbf{u}(\mathbf{x}_b, t') \cdot \nabla p(\mathbf{x}_b, t') \right] dt'$$

Note that the function being integrated in the integral defining $K(\mathbf{x}_b, t_m)$ is smooth over the entire integration interval $[0, t_m]$, and this interval is finite. Therefore, $K(\mathbf{x}_b, t_m)$ must be finite. Differentiating equation (3.104) with respect to t then gives us

$$\frac{dK_b}{dt} \;=\; \nu \nabla^2 K(\mathbf{x}_b, t) - \nu \sum_{i=1}^3 |\nabla u_i(\mathbf{x}_b, t)|^2 \tag{3.105}$$
$$- \; \mathbf{u}(\mathbf{x}_b, t) \cdot \nabla K(\mathbf{x}_b, t) - \mathbf{u}(\mathbf{x}_b, t) \cdot \nabla p(\mathbf{x}_b, t)$$

Now since $\mathbf{x}_b$ is a relative maximum of K, and K is a smooth function, we must have $\nabla K(\mathbf{x}_b, t) = 0$, and the third term on the right-hand side of equation (3.105) vanishes. Therefore, equation (3.105) becomes

$$\frac{dK_b}{dt} \;=\; \nu \nabla^2 K(\mathbf{x}_b, t) - \nu \sum_{i=1}^3 |\nabla u_i(\mathbf{x}_b, t)|^2 - \mathbf{u}(\mathbf{x}_b, t) \cdot \nabla p(\mathbf{x}_b, t)$$

Also, we must have $\nabla^2 K(\mathbf{x}_b, t) \leq 0$. This comes from the "second derivative test" for a relative maximum or minimum at a stationary point in a smooth function. The second term on the right-hand side is also negative or zero since $|\nabla u_i|^2 \geq 0$. Therefore, equation (3.105) implies that

$$\frac{dK_b}{dt}(t) \;\leq\; - \, \mathbf{u}(\mathbf{x}_b, t) \cdot \nabla p(\mathbf{x}_b, t)$$

Now, the maximum value that can be obtain on the right-hand side of this inequality occurs for the case where $\mathbf{u}(\mathbf{x}_b, t)$ and $\nabla p(\mathbf{x}_b, t)$ are anti-parallel vectors. In this case, we have

$$\frac{dK_b}{dt}(t) \;\leq\; |\mathbf{u}(\mathbf{x}_b, t)||\nabla p(\mathbf{x}_b, t)| \;=\; \sqrt{2}\,|\nabla p(\mathbf{x}_b, t)|\,\sqrt{K_b(t)} \tag{3.106}$$

At this point, we define the function $K_b^*(t)$ as the maximum value of $K_b(t)$ allowed by inequality (3.106) for $t_m \leq t < t_b$. Therefore, we have

$$\frac{dK_b^*}{dt} = \sqrt{2}\,|\nabla p(\mathbf{x}_b, t)|\,\sqrt{K_b^*(t)} \qquad (3.107)$$

To solve this equation, we divide both sides by $\sqrt{K_b^*}$ and integrate with respect to t to obtain

$$K_b^*(t) = \frac{1}{2}\left(\int_{t_m}^{t} |\nabla p(\mathbf{x}_b, t')|\,dt'\right)^2 \qquad (3.108)$$

From inequality (3.101), we see that the time integral in this equation is finite for all values of $t > t_m$ and in fact has a finite limit as $t \to t_b$. Therefore, from equation (3.103), we have

$$\lim_{t \to t_b} K(\mathbf{x}_b, t) = K(\mathbf{x}_b, t_m) + \lim_{t \to t_b} K_b(t) = \text{finite} \qquad (3.109)$$

Hence, the conjectured smooth blowup cannot occur, and we have shown existence and smoothness of the solution $[\mathbf{u}(\mathbf{x}, t),\, p(\mathbf{x}, t)]$ for all $\mathbf{x} \in \mathbb{R}^3$ and $t \geq 0$, and the existence of solution part of our main theorem is proven.

3.5 Uniqueness of Solution. Let us now show that the solution of the given problem is in fact unique. We start by defining $\mathbf{u}^{(1)}(\mathbf{x}, t)$ and $\mathbf{u}^{(2)}(\mathbf{x}, t)$ along with the corresponding scalar pressure functions $p^{(1)}(\mathbf{x}, t)$ and $p^{(2)}(\mathbf{x}, t)$ as two possible solutions of equation (2.4) with initial condition (2.5) and zero-divergence constraint (2.9). We therefore write

$$\frac{\partial \mathbf{u}^{(1)}}{\partial t} = \nu \nabla^2 \mathbf{u}^{(1)} - (\mathbf{u}^{(1)} \cdot \nabla)\mathbf{u}^{(1)} - \nabla p^{(1)} \qquad (3.110)$$

and

$$\frac{\partial \mathbf{u}^{(2)}}{\partial t} = \nu \nabla^2 \mathbf{u}^{(2)} - (\mathbf{u}^{(2)} \cdot \nabla)\mathbf{u}^{(2)} - \nabla p^{(2)} \qquad (3.111)$$

Subtracting equation (3.110) from (3.111), we have

$$\frac{\partial \mathbf{D}}{\partial t} = \nu \nabla^2 \mathbf{D} - (\mathbf{u}^{(2)} \cdot \nabla)\mathbf{D} - (\mathbf{D} \cdot \nabla)\mathbf{u}^{(1)} + \nabla p^{(1)} - \nabla p^{(2)} \qquad (3.112)$$

where we have defined

$$\mathbf{D}(\mathbf{x}, t) = \mathbf{u}^{(2)}(\mathbf{x}, t) - \mathbf{u}^{(1)}(\mathbf{x}, t) \qquad (3.113)$$

as the difference between the two solutions. Uniqueness will be proven if we can show that $\mathbf{D}(\mathbf{x}, t) = 0$ for all $\mathbf{x} \in \mathbb{R}^3$ and $t \geq 0$.

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Taking the scalar product of both sides of equation (3.112) with $\mathbf{D}$, we have

$$\mathbf{D} \cdot \frac{\partial \mathbf{D}}{\partial t} = \nu \mathbf{D} \cdot \nabla^2 \mathbf{D} - \mathbf{D} \cdot \left[(\mathbf{u}^{(2)} \cdot \nabla) \mathbf{D} \right] - \mathbf{D} \cdot \left[(\mathbf{D} \cdot \nabla) \mathbf{u}^{(1)} \right] + \mathbf{D} \cdot \nabla p^{(1)} - \mathbf{D} \cdot \nabla p^{(2)}$$

$$= \nu \sum_{i=1}^{3} D_i \nabla^2 D_i - \sum_{i=1}^{3} \sum_{k=1}^{3} D_i u_k^{(2)} \frac{\partial D_i}{\partial x_k} - \sum_{i=1}^{3} \sum_{k=1}^{3} D_i D_k \frac{\partial u_i^{(1)}}{\partial x_k} - \mathbf{D} \cdot (\nabla p^{(2)} - \nabla p^{(1)})$$

$$= \nu \sum_{i=1}^{3} \nabla \cdot (D_i \nabla D_i) - \nu \sum_{i=1}^{3} (\nabla D_i) \cdot (\nabla D_i) - \mathbf{u}^{(2)} \cdot \nabla \left(\frac{1}{2} \mathbf{D} \cdot \mathbf{D} \right) \qquad (3.114)$$

$$\quad - \mathbf{D} \cdot \left[(\mathbf{D} \cdot \nabla) \mathbf{u}^{(1)} \right] - \mathbf{D} \cdot (\nabla p^{(2)} - \nabla p^{(1)})$$

$$= \nu \sum_{i=1}^{3} \nabla \cdot (D_i \nabla D_i) - \nu \sum_{i=1}^{3} (\nabla D_i) \cdot (\nabla D_i) - \nabla \cdot \left(\frac{1}{2} (\mathbf{D} \cdot \mathbf{D}) \, \mathbf{u}^{(2)} \right) + \frac{1}{2} (\mathbf{D} \cdot \mathbf{D}) \, \nabla \cdot \mathbf{u}^{(2)}$$

$$\quad - \mathbf{D} \cdot \left[(\mathbf{D} \cdot \nabla) \mathbf{u}^{(1)} \right] - \nabla \cdot \left[(p^{(2)} - p^{(1)}) \mathbf{D} \right] + (p^{(2)} - p^{(1)}) \, \nabla \cdot \mathbf{D}$$

Since $\nabla \cdot \mathbf{u}^{(2)} = 0$ and $\nabla \cdot \mathbf{D} = 0$, the fourth and seventh terms on the right-hand side of this equation vanish, and we write

$$\frac{\partial W_D}{\partial t} = \nu \sum_{i=1}^{3} \nabla \cdot (D_i \nabla D_i) - \nu \sum_{i=1}^{3} (\nabla D_i) \cdot (\nabla D_i) \qquad (3.115)$$

$$\quad - \nabla \cdot \left(W_D \mathbf{u}^{(2)} \right) - \mathbf{D} \cdot \left[(\mathbf{D} \cdot \nabla) \mathbf{u}^{(1)} \right] - \nabla \cdot (p_D \mathbf{D})$$

where we have defined the normalized energy density W_D associated with $\mathbf{D}$, and pressure difference p_D as

$$W_D = \frac{1}{2} (\mathbf{D} \cdot \mathbf{D}) \qquad \text{and} \qquad p_D = p^{(2)} - p^{(1)} \qquad (3.116),(3.117)$$

Integrating equation (3.115) over all $\mathbb{R}^3$ space, we obtain

$$\frac{dE_D}{dt} = \nu \sum_{i=1}^{3} \int_{\mathbb{R}^3} \nabla \cdot (D_i \nabla D_i) \, d^3\mathbf{x} - \nu \int_{\mathbb{R}^3} \sum_{i=1}^{3} (\nabla D_i) \cdot (\nabla D_i) \, d^3\mathbf{x} \qquad (3.118)$$

$$\quad - \int_{\mathbb{R}^3} \nabla \cdot \left(W_D \mathbf{u}^{(2)} \right) d^3\mathbf{x} - \int_{\mathbb{R}^3} \mathbf{D} \cdot \left[(\mathbf{D} \cdot \nabla) \mathbf{u}^{(1)} \right] d^3\mathbf{x} - \int_{\mathbb{R}^3} \nabla \cdot (p_D \mathbf{D}) \, d^3\mathbf{x}$$

where we have defined the normalized total energy density associated with $\mathbf{D}$ as

$$E_D(t) = \int_{\mathbb{R}^3} W_D(\mathbf{x}, t) \, d^3\mathbf{x} = \frac{1}{2} \int_{\mathbb{R}^3} \mathbf{D}(\mathbf{x}, t) \cdot \mathbf{D}(\mathbf{x}, t) \, d^3\mathbf{x} \qquad (3.119)$$

The first, third, and fifth terms on the right-hand side of equation (3.118) vanish via the divergence theorem and the fact that the integrands in each of these terms approach zero as $1/(|\mathbf{x}| + a)^{2\kappa}$ as $|\mathbf{x}| \to \infty$. Therefore, equation (3.118) becomes

$$\frac{dE_D}{dt} = - \int_{\mathbb{R}^3} \left(\sum_{i=1}^{3} \nu (\nabla D_i) \cdot (\nabla D_i) + \mathbf{D} \cdot \left[(\mathbf{D} \cdot \nabla) \mathbf{u}^{(1)} \right] \right) d^3\mathbf{x} = Y(t) \qquad (3.120)$$

where we have defined

$$Y(t) = - \int_{\mathbb{R}^3} \left(\sum_{i=1}^{3} \nu (\nabla D_i) \cdot (\nabla D_i) + \mathbf{D} \cdot \left[(\mathbf{D} \cdot \nabla) \mathbf{u}^{(1)} \right] \right) d^3\mathbf{x} \qquad (3.121)$$

At this point, we can show that $E_D(t) = 0$ and $Y(t) = 0$ for all times t using a simple finite difference approach. We first define a set of N discrete time values t'_n

such that

$$0 = t'_0 < t'_1 < t'_2 < \ ... \ < t'_N = t \tag{3.122}$$

Let us suppose that $\mathbf{D}(\mathbf{x}, t'_n) = 0$ for all $\mathbf{x} \in \mathbb{R}^3$ at some time step n. Then, according to equations (3.119) and (3.121), we would have $E_D(t'_n) = 0$ and $Y(t'_n) = 0$. Applying the finite difference method at this time step, we write $E_D(t'_{n+1})$ at the next time step as

$$\begin{aligned} E_D(t'_{n+1}) &= E_D(t'_n) + \frac{dE_D}{dt}(t'_n)(t'_{n+1} - t'_n) \\ &= E_D(t'_n) + Y(t'_n)(t'_{n+1} - t'_n) \end{aligned} \tag{3.123}$$

where we have used equation (3.120) in the last step. But since $\mathbf{D}(\mathbf{x}, t'_n) = 0$ for all $\mathbf{x} \in \mathbb{R}^3$, we must have $Y(t'_n) = 0$ according to equation (3.121). Therefore, equation (3.123) reduces to

$$E_D(t'_{n+1}) = E_D(t'_n) = 0 \tag{3.124}$$

and since $\mathbf{D}(\mathbf{x}, t)$ is a smooth function, it then follows from equation (3.119) that $\mathbf{D}(\mathbf{x}, t'_{n+1}) = 0$ for all $\mathbf{x} \in \mathbb{R}^3$. Hence, we have shown that if $\mathbf{D}(\mathbf{x}, t'_n) = 0$ for all $\mathbf{x} \in \mathbb{R}^3$, then the same must be true for $\mathbf{D}(\mathbf{x}, t'_{n+1})$. From this, it follows inductively that since $\mathbf{D}(\mathbf{x}, 0) = \mathbf{D}(\mathbf{x}, t'_0) = 0$ for all $\mathbf{x} \in \mathbb{R}^3$, we must have

$$\mathbf{D}(\mathbf{x}, t'_n) = 0 \quad \text{for} \quad 0 \le n \le N \ \text{ and } \ \mathbf{x} \in \mathbb{R}^3 \tag{3.125}$$

and therefore $\mathbf{D}(\mathbf{x}, t) = \mathbf{D}(\mathbf{x}, t'_N) = 0$. Since this holds for all values of $\Delta t'_n = t'_{n+1} - t'_n$, it must also hold in the limit as the $\Delta t'_n \to 0$. Therefore, we have shown that $\mathbf{D}(\mathbf{x}, t) = 0$ for all $\mathbf{x} \in \mathbb{R}^3$ and $t \ge 0$. Hence, the uniqueness part of the main theorem is proven.

4.0 Conclusion

In this paper, we have shown existence and smoothness of a solution to the zero driving-force Navier-Stokes equation in free space with given initial fluid velocity and spatial derivatives profiles which approach zero as $a^\kappa/(|\mathbf{x}| + a)^\kappa$ as $|\mathbf{x}| \to \infty$, assuming a scalar pressure and incompressibility of the fluid. While the existence of such a solution initially was a given, the continuation of this solution for all times $t \ge 0$ needed to be proven. First, it was shown that the solution $\mathbf{u}(\mathbf{x}, t)$ must remain smooth for as long as it does exist (ie. remains finite). Therefore, if $\mathbf{u}(\mathbf{x}, t)$ becomes infinitely large in finite time, it must be a "smooth blowup" where a global maximum first forms at the blowup point, and the fluid velocity reaches infinite values at this point. It was shown, however, that this smooth blowup could not occur since at a maximum point, the Navier-Stokes equation predicts that the only force that can provide positive acceleration to the fluid element is the scalar pressure gradient. The maximum total increase in fluid velocity provided by this force is given by the time integral of $|\nabla p(\mathbf{x}_b, t)|$ after formation of the global maximum. This integral, however, was shown to be finite for all times that the fluid velocity remained smooth, and therefore a smooth blowup is not possible. Finally, we showed that the solution for a given initial fluid velocity $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x})$ is unique.

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