Proposed Counter-Example

In this proposed counter-example to statement that equations (108) and (109) of the paper imply equation (118), we choose the quantity $q(\mathbf{x},t)$ as follows:

$$q(\mathbf{x},t) = q_0\,\frac{\delta_0^3}{\delta^3(t)} \quad \text{if } |\mathbf{x}| < \delta(t), \quad \text{and} \quad q(\mathbf{x},t) = 0 \quad \text{if } |\mathbf{x}| \geq \delta(t)$$

where

$$\delta(t) = \delta_0\,\frac{t_b - t}{t_b},$$

t_b is the blowup time, and q_0 and δ_0 are the initial values of q and δ respectively. Integrating this function over $\mathbb{R}^3$, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} q(\mathbf{x},t)\,d^3\mathbf{x} \;&=\; \int_0^{\delta(t)}\int_0^{\pi}\int_0^{2\pi} q_0\,\frac{\delta_0^3}{\delta^3(t)}\,r^2\sin\theta\,d\phi\,d\theta\,dr \;=\; 4\pi q_0\,\frac{\delta_0^3}{\delta^3(t)}\int_0^{\delta(t)} r^2\,dr \\
&=\; \frac{4}{3}\pi q_0\,\frac{\delta_0^3}{\delta^3(t)}\,\delta^3(t) \;=\; \frac{4}{3}\pi q_0\,\delta_0^3 \;=\; \text{finite}
\end{aligned}$$

which is consistent with inequality (109) in the paper.

Now let us see if we can establish existence of the time integral of $|\nabla p|$ at $\mathbf{x} = 0$. We first obtain an expression for the function $Q(\mathbf{x},t)$ valid for $|\mathbf{x}| < \delta(t)$. From equation (108) of the paper, we have

$$|Q(\mathbf{x},t)| \;=\; \frac{\partial q}{\partial t}(\mathbf{x},t) \;=\; 3q_0\,\frac{\delta_0^4}{t_b\delta^4(t)} \;=\; Q(\mathbf{x},t)$$

where we have assumed $Q(\mathbf{x},t) \geq 0$ for this example. As we will see, however, using the opposite sign leads to the same basic result. Inserting this function into equation (12) from the paper gives us

$$\nabla^2 p(\mathbf{x},t) \;=\; -Q(\mathbf{x},t) \;=\; -3q_0\,\frac{\delta_0^4}{t_b\delta^4(t)} \quad \text{if } |\mathbf{x}| < \delta(t); \quad \text{and} \quad 0 \text{ otherwise}$$

This is merely Poisson's equation with a radially symmetric non-homogeneous term. Therefore we must have $\nabla p = 0$ at $\mathbf{x} = 0$ for all $t < t_b$. Note that we would have gotten the same result $\nabla p = 0$ if we had chosen $Q(\mathbf{x},t) \leq 0$ at $\mathbf{x} = 0$. The mathematical principles are the same as those used to show that a radially symmetric charge distribution has zero electric field at the center, and that there is zero gravitational field at the center of a star or planet. Therefore, we must have

$$\Lambda(0,t) \;=\; \int_0^t |\nabla p(0,t')|\,dt' \;=\; 0$$

for all $t < t_b$. Since $\Lambda(0,t)$ does not blow up as $t \to t_b$, this proposed counter-example to the claim that (108) and (109) implies (118) is false.