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EXISTENCE, UNIQUENESS AND SMOOTHNESS OF A SOLUTION FOR 3D NAVIER-STOKES EQUATIONS WITH ANY SMOOTH INITIAL VELOCITY

ARKADIY TSIONSKIY, MIKHAIL TSIONSKIY

Abstract. Solutions of the Navier-Stokes and Euler equations with initial conditions for 2D and 3D cases were obtained in the form of converging series, by an analytical iterative method using Fourier and Laplace transforms in [28, 29]. There the solutions are infinitely differentiable functions, and for several combinations of parameters numerical results are presented. This article provides a detailed proof of the existence, uniqueness and smoothness of the solution of the Cauchy problem for the 3D Navier-Stokes equations with any smooth initial velocity. When the viscosity tends to zero, this proof applies also to the Euler equations.

1. Introduction

Many authors have obtained results regarding the Euler and Navier-Stokes equations. Existence and smoothness of solution for the Navier-Stokes equations in two dimensions have been known for a long time. Leray (1934) showed that the Navier-Stokes equations in three dimensional space have a weak solution. Scheffer (1976, 1993) and Shnirelman (1997) obtained weak solution of the Euler equations with compact support in space-time. Caffarelli, Kohn and Nirenberg (1982) improved Scheffer's results, and Lin (1998) simplified the proof of the results by Leray. Many problems and conjectures about behavior of weak solutions of the Euler and Navier-Stokes equations are described in the books by Ladyzhenskaya (1969), Temam (1977), Constantin (2001), Bertozzi and Majda (2002) or Lemarié-Rieusset (2002).

The solution of the Cauchy problem for the 3D Navier-Stokes equations is described in this article. We will consider an initial velocity that is infinitely differentiable and decreasing rapidly to zero in infinity. The applied force is assumed to be identically zero. A solution of the problem will be presented in the following stages:

First stage (sections 2, 3). We move the non-linear parts of equations to the right side. Then we solve the system of linear partial differential equations with constant coefficients. We have obtained the solution of this system using Fourier

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transforms for the space coordinates and Laplace transform for time. From theorems about applications of Fourier and Laplace transforms, for system of linear partial differential equations with constant coefficients, we see that in this case if initial velocity and applied force are smooth enough functions decreasing in infinity, then the solution of such system is also a smooth function. Corresponding theorems are presented in Bochner [3], Palamodov [18], Shilov [23], Hormander [9], Mizohata [17], Treves [27]. The result of this stage is an integral equation for the vector-function of velocity.

Second stage (sections 4, 5). We introduce perfect spaces of functions and vector-functions (Gel'fand, Chilov [7]), in which we look for the solution of the problem. We demonstrate the equivalence of solving the Cauchy problem in differential form and in the form of an integral equation.

Third stage (section 6). We divide all parts of the integral equation by an appropriate constant depending on value of initial fluid velocity, and obtain the equivalent integral equation. We also replace the corresponding integration variables in the integral operators. This newly received equivalent integral equation allowed us to analyze the Cauchy problem for the 3D Navier-Stokes equations for any value of initial fluid velocity. According to a priori estimate of the solution of the Cauchy problem for the 3D Navier-Stokes equations [13, 12], the described constant is proportional to max of the norms of the initial velocity in the spaces C^2 and L_2.

Fourth stage (section 6). We use the newly obtained equivalent integral equation to prove the existence and uniqueness of the solution of the Cauchy problem in the time range $[0, \infty)$ based on the Caccioppoli-Banach fixed point theorem (Kantorovich, Akilov [10], Trenogin [26], Rudin [20], Kirk and Sims [11], Granas and Dugundji [8], Ayerbe Toledano, Dominguez Benavides, Lopez Acedo [1]). For this purpose the following three theorems are proven in this article: Theorem 6.1: the integral operator of the problem is a contraction operator; Theorem 6.2: the existence and uniqueness of the solution of the problem is valid for any $t \in [0, \infty)$; Theorem 6.3: the solution of the problem depends continuously on t.

Fifth stage (section 6). By using a priori estimate of the solution of the Cauchy problem for the 3D Navier-Stokes equations [13, 12], we show that the energy of the whole process has a finite value for any t in $[0, \infty)$.

2. Mathematical setup

The Navier-Stokes equations describe the motion of a fluid in $\mathbb{R}^N$ $(N = 3)$. We look for a viscous incompressible fluid filling all of $\mathbb{R}^N$ here. The Navier-Stokes equations are then given by

$$\frac{\partial u_k}{\partial t} + \sum_{n=1}^{N} u_n \frac{\partial u_k}{\partial x_n} = \nu \Delta u_k - \frac{\partial p}{\partial x_k} + f_k(x,t) \quad x \in \mathbb{R}^N,\ t \geq 0,\ 1 \leq k \leq N\,, \tag{2.1}$$

$$\operatorname{div} \vec{u} = \sum_{n=1}^{N} \frac{\partial u_n}{\partial x_n} = 0 \quad x \in \mathbb{R}^N,\ t \geq 0\,, \tag{2.2}$$

with initial conditions

$$\vec{u}(x,0) = \vec{u}^0(x) \quad x \in \mathbb{R}^N\,. \tag{2.3}$$

Here $\vec{u}(x,t) = (u_k(x,t)) \in \mathbb{R}^N$ $(1 \leq k \leq N)$ is an unknown velocity vector, $N = 3$; $p(x,t)$ is an unknown pressure; $\vec{u}^0(x)$ is a given C^∞ divergence-free vector field;

$f_k(x,t)$ are components of a given, externally applied force $\vec{f}(x,t)$; ν is a positive coefficient of the viscosity (if $\nu = 0$ then (2.1)–(2.3) are the Euler equations); and $\Delta = \sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2}$ is the Laplacian in the space variables. Equation (2.1) is Newton's law for a fluid element. Equation (2.2) says that the fluid is incompressible. For physically reasonable solutions, we accept

$$u_k(x,t) \to 0, \quad \frac{\partial u_k}{\partial x_n} \to 0 \quad \text{as } |x| \to \infty \quad 1 \le k \le N,\ 1 \le n \le N\,. \tag{2.4}$$

Hence, we will restrict attention to initial conditions $\vec{u}^0$ and force $\vec{f}$ that satisfy

$$|\partial_x^\alpha \vec{u}^0(x)| \le C_{\alpha K}(1+|x|)^{-K} \quad \text{on } \mathbb{R}^N \text{ for any } \alpha \text{ and any } K. \tag{2.5}$$

and

$$|\partial_x^\alpha \partial_t^\beta \vec{f}(x,t)| \le C_{\alpha\beta K}(1+|x|+t)^{-K} \quad \text{on } \mathbb{R}^N \times [0,\infty) \text{ for any } \alpha, \beta \text{ and any } K. \tag{2.6}$$

To start the process of solution let us add $-\sum_{n=1}^{N} u_n \frac{\partial u_k}{\partial x_n}$ to both sides of the equations (2.1). Then we have

$$\frac{\partial u_k}{\partial t} = \nu\,\Delta\, u_k - \frac{\partial p}{\partial x_k} + f_k(x,t) - \sum_{n=1}^{N} u_n \frac{\partial u_k}{\partial x_n} \quad x \in \mathbb{R}^N,\ t \ge 0,\ 1 \le k \le N, \tag{2.7}$$

$$\operatorname{div} \vec{u} = \sum_{n=1}^{N} \frac{\partial u_n}{\partial x_n} = 0 \quad x \in \mathbb{R}^N,\ t \ge 0, \tag{2.8}$$

$$\vec{u}(x,0) = \vec{u}^0(x) \quad x \in \mathbb{R}^N, \tag{2.9}$$

$$u_k(x,t) \to 0 \quad \frac{\partial u_k}{\partial x_n} \to 0 \quad \text{as } |x| \to \infty \quad 1 \le k \le N,\ 1 \le n \le N, \tag{2.10}$$

$$|\partial_x^\alpha \vec{u}^0(x)| \le C_{\alpha K}(1+|x|)^{-K} \quad \text{on } \mathbb{R}^N \text{ for any } \alpha \text{ and any } K, \tag{2.11}$$

$$|\partial_x^\alpha \partial_t^\beta \vec{f}(x,t)| \le C_{\alpha\beta K}(1+|x|+t)^{-K} \quad \text{on } \mathbb{R}^N \times [0,\infty) \text{ for any } \alpha, \beta \text{ and any } K. \tag{2.12}$$

Let us denote

$$\tilde{f}_k(x,t) = f_k(x,t) - \sum_{n=1}^{N} u_n \frac{\partial u_k}{\partial x_n} \quad 1 \le k \le N\,. \tag{2.13}$$

We can present it in the vector form as

$$\vec{\tilde{f}}(x,t) = \vec{f}(x,t) - (\vec{u} \cdot \nabla)\vec{u}\,. \tag{2.14}$$

3. Solution of the system (2.7)–(2.14)

Let us assume that all operations below are valid. The validity of these operations will be proved in the next sections. Taking into account our substitution (2.13) we see that (2.7)–(2.9) are in fact system of linear partial differential equations with constant coefficients.

The solution of this system will be presented by the following steps:

First step. We use Fourier transform (7.1) to solve equations (2.7)–(2.14). We obtain:

$$U_k(\gamma_1, \gamma_2, \gamma_3, t) = F[u_k(x_1, x_2, x_3, t)],$$

$$\begin{gathered}
-\gamma_s^2 U_k(\gamma_1,\gamma_2,\gamma_3,t) = F[\frac{\partial^2 u_k(x_1,x_2,x_3,t)}{\partial x_s^2}] \quad \text{(use (2.10))}, \\
U_k^0(\gamma_1,\gamma_2,\gamma_3) = F[u_k^0(x_1,x_2,x_3)], \\
P(\gamma_1,\gamma_2,\gamma_3,t) = F[p(x_1,x_2,x_3,t)], \\
\tilde{F}_k(\gamma_1,\gamma_2,\gamma_3,t) = F[\tilde{f}_k(x_1,x_2,x_3,t)],
\end{gathered}$$

for $k,s = 1,2,3$. Then

$$\begin{aligned}
\frac{dU_1(\gamma_1,\gamma_2,\gamma_3,t)}{dt} &= -\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)U_1(\gamma_1,\gamma_2,\gamma_3,t) + i\gamma_1 P(\gamma_1,\gamma_2,\gamma_3,t) \\
&\quad + \tilde{F}_1(\gamma_1,\gamma_2,\gamma_3,t),
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
\frac{dU_2(\gamma_1,\gamma_2,\gamma_3,t)}{dt} &= -\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)U_2(\gamma_1,\gamma_2,\gamma_3,t) + i\gamma_2 P(\gamma_1,\gamma_2,\gamma_3,t) \\
&\quad + \tilde{F}_2(\gamma_1,\gamma_2,\gamma_3,t),
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\frac{dU_3(\gamma_1,\gamma_2,\gamma_3,t)}{dt} &= -\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)U_3(\gamma_1,\gamma_2,\gamma_3,t) + i\gamma_3 P(\gamma_1,\gamma_2,\gamma_3,t) \\
&\quad + \tilde{F}_3(\gamma_1,\gamma_2,\gamma_3,t),
\end{aligned} \tag{3.3}$$

$$\gamma_1 U_1(\gamma_1,\gamma_2,\gamma_3,t) + \gamma_2\, U_2(\gamma_1,\gamma_2,\gamma_3,t) + \gamma_3\, U_3(\gamma_1,\gamma_2,\gamma_3,t) = 0, \tag{3.4}$$

$$U_1(\gamma_1,\gamma_2,\gamma_3,0) = U_1^0(\gamma_1,\gamma_2,\gamma_3), \tag{3.5}$$

$$U_2(\gamma_1,\gamma_2,\gamma_3,0) = U_2^0(\gamma_1,\gamma_2,\gamma_3), \tag{3.6}$$

$$U_3(\gamma_1,\gamma_2,\gamma_3,0) = U_3^0(\gamma_1,\gamma_2,\gamma_3)\,. \tag{3.7}$$

Hence, we have received a system of linear ordinary differential equations with constant coefficients (3.1)-(3.7) according to Fourier transforms. At the same time the initial conditions are set only for Fourier transforms of velocity components $U_1(\gamma_1,\gamma_2,\gamma_3,t)$, $U_2(\gamma_1,\gamma_2,\gamma_3,t)$, $U_3(\gamma_1,\gamma_2,\gamma_3,t)$. Because of that we can eliminate Fourier transform for pressure $P(\gamma_1,\gamma_2,\gamma_3,t)$ from equations (3.1)–(3.3) on the next step of the solution process.

Second step. From here assuming that $\gamma_1 \neq 0, \gamma_2 \neq 0, \gamma_3 \neq 0$, we eliminate $P(\gamma_1,\gamma_2,\gamma_3,t)$ from equations (3.1) – (3.3) and find

$$\begin{aligned}
&\frac{d}{dt}[U_2(\gamma_1,\gamma_2,\gamma_3,t) - \frac{\gamma_2}{\gamma_1}\, U_1(\gamma_1,\gamma_2,\gamma_3,t)] \\
&= -\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)[U_2(\gamma_1,\gamma_2,\gamma_3,t) - \frac{\gamma_2}{\gamma_1} U_1(\gamma_1,\gamma_2,\gamma_3,t)] \\
&\quad + [\tilde{F}_2(\gamma_1,\gamma_2,\gamma_3,t) - \frac{\gamma_2}{\gamma_1}\tilde{F}_1(\gamma_1,\gamma_2,\gamma_3,t)]\,,
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&\frac{d}{dt}[U_3(\gamma_1,\gamma_2,\gamma_3,t) - \frac{\gamma_3}{\gamma_1}\, U_1(\gamma_1,\gamma_2,\gamma_3,t)] \\
&= -\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)[U_3(\gamma_1,\gamma_2,\gamma_3,t) - \frac{\gamma_3}{\gamma_1}\, U_1(\gamma_1,\gamma_2,\gamma_3,t)] \\
&\quad + [\tilde{F}_3(\gamma_1,\gamma_2,\gamma_3,t) - \frac{\gamma_3}{\gamma_1}\tilde{F}_1(\gamma_1,\gamma_2,\gamma_3,t)]\,,
\end{aligned} \tag{3.9}$$

$$\gamma_1 U_1(\gamma_1,\gamma_2,\gamma_3,t) + \gamma_2\, U_2(\gamma_1,\gamma_2,\gamma_3,t) + \gamma_3\, U_3(\gamma_1,\gamma_2,\gamma_3,t) = 0, \tag{3.10}$$

$$U_1(\gamma_1,\gamma_2,\gamma_3,0) = U_1^0(\gamma_1,\gamma_2,\gamma_3), \tag{3.11}$$

$$U_2(\gamma_1,\gamma_2,\gamma_3,0) = U_2^0(\gamma_1,\gamma_2,\gamma_3), \tag{3.12}$$

$$U_3(\gamma_1,\gamma_2,\gamma_3,0)=U_3^0(\gamma_1,\gamma_2,\gamma_3)\,. \tag{3.13}$$

Third step. We use Laplace transform (7.2), (7.3) for a system of linear ordinary differential equations with constant coefficients (3.8)–(3.10) and have as a result the system of linear algebraic equations with constant coefficients:

$$U_k^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)=L[U_k(\gamma_1,\gamma_2,\gamma_3,t)]\quad k=1,2,3; \tag{3.14}$$

$$\tilde F_k^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)=L[\tilde F_k(\gamma_1,\gamma_2,\gamma_3,t)]\quad k=1,2,3; \tag{3.15}$$

$$\begin{aligned}
&\eta[U_2^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)-\frac{\gamma_2}{\gamma_1}U_1^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)]\\
&\quad-[U_2(\gamma_1,\gamma_2,\gamma_3,0)-\frac{\gamma_2}{\gamma_1}U_1(\gamma_1,\gamma_2,\gamma_3,0)]\\
&=-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)[U_2^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)-\frac{\gamma_2}{\gamma_1}U_1^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)]\\
&\quad+[\tilde F_2^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)-\frac{\gamma_2}{\gamma_1}\tilde F_1^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)],
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\eta[U_3^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)-\frac{\gamma_3}{\gamma_1}U_1^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)]\\
&\quad-[U_3(\gamma_1,\gamma_2,\gamma_3,0)-\frac{\gamma_3}{\gamma_1}U_1(\gamma_1,\gamma_2,\gamma_3,0)]\\
&=-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)[U_3^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)-\frac{\gamma_3}{\gamma_1}U_1^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)]\\
&\quad+[\tilde F_3^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)-\frac{\gamma_3}{\gamma_1}\tilde F_1^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)],
\end{aligned} \tag{3.17}$$

$$\gamma_1U_1^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)+\gamma_2\,U_2^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)+\gamma_3\,U_3^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)=0, \tag{3.18}$$

$$U_1(\gamma_1,\gamma_2,\gamma_3,0)=U_1^0(\gamma_1,\gamma_2,\gamma_3), \tag{3.19}$$

$$U_2(\gamma_1,\gamma_2,\gamma_3,0)=U_2^0(\gamma_1,\gamma_2,\gamma_3), \tag{3.20}$$

$$U_3(\gamma_1,\gamma_2,\gamma_3,0)=U_3^0(\gamma_1,\gamma_2,\gamma_3)\,. \tag{3.21}$$

Let us rewrite system of equations (3.16)–(3.18) in the form

$$\begin{aligned}
&[\eta+\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)]\frac{\gamma_2}{\gamma_1}U_1^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)\\
&\quad-[\eta+\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)]U_2^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)\\
&=[\frac{\gamma_2}{\gamma_1}\tilde F_1^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)-\tilde F_2^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)]\\
&\quad+[\frac{\gamma_2}{\gamma_1}U_1(\gamma_1,\gamma_2,\gamma_3,0)-U_2(\gamma_1,\gamma_2,\gamma_3,0)],
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
&[\eta+\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)]\frac{\gamma_3}{\gamma_1}U_1^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)\\
&\quad-[\eta+\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)]U_3^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)\\
&=[\frac{\gamma_3}{\gamma_1}\tilde F_1^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)-\tilde F_3^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)]\\
&\quad+[\frac{\gamma_3}{\gamma_1}U_1(\gamma_1,\gamma_2,\gamma_3,0)-U_3(\gamma_1,\gamma_2,\gamma_3,0)],
\end{aligned} \tag{3.23}$$

$$\gamma_1U_1^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)+\gamma_2\,U_2^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)+\gamma_3\,U_3^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)=0 \tag{3.24}$$

The determinant of this system is

$$\begin{aligned}
\Delta &= \begin{vmatrix} [\eta+\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)]\frac{\gamma_2}{\gamma_1} & -[\eta+\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)] & 0 \\ [\eta+\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)]\frac{\gamma_3}{\gamma_1} & 0 & -[\eta+\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)] \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \\
&= \frac{[\eta+\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)]^2(\gamma_1^2+\gamma_2^2+\gamma_3^2)}{\gamma_1} \neq 0\,.
\end{aligned} \tag{3.25}$$

Consequently the system of equations (3.16)–(3.18) and/or (3.22)–(3.24) has a unique solution. Taking into account formulas (3.19)–(3.21) we can write this solution in the form

$$\begin{aligned}
&U_1^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta) \\
&= \frac{[(\gamma_2^2+\gamma_3^2)\tilde F_1^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta) - \gamma_1\gamma_2\tilde F_2^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta) - \gamma_1\gamma_3\tilde F_3^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)]}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)[\eta+\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)]} \\
&\quad + \frac{U_1^0(\gamma_1,\gamma_2,\gamma_3)}{[\eta+\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)]}\,,
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
&U_2^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta) \\
&= \frac{[(\gamma_3^2+\gamma_1^2)\tilde F_2^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta) - \gamma_2\gamma_3\tilde F_3^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta) - \gamma_2\gamma_1\tilde F_1^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)]}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)[\eta+\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)]} \\
&\quad + \frac{U_2^0(\gamma_1,\gamma_2,\gamma_3)}{[\eta+\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)]}\,,
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
&U_3^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta) \\
&= \frac{[(\gamma_1^2+\gamma_2^2)\tilde F_3^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta) - \gamma_3\gamma_1\tilde F_1^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta) - \gamma_3\gamma_2\tilde F_2^{\otimes}(\gamma_1,\gamma_2,\gamma_3,\eta)]}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)[\eta+\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)]} \\
&\quad + \frac{U_3^0(\gamma_1,\gamma_2,\gamma_3)}{[\eta+\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)]}\,.
\end{aligned} \tag{3.28}$$

Then we use the convolution theorem with the convolution formula (7.4) and integral (7.5) for (3.26)–(3.28) to obtain

$$\begin{aligned}
&U_1(\gamma_1,\gamma_2,\gamma_3,t) \\
&= \int_0^t e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)(t-\tau)} \\
&\quad \times \frac{[(\gamma_2^2+\gamma_3^2)\tilde F_1(\gamma_1,\gamma_2,\gamma_3,\tau) - \gamma_1\gamma_2\tilde F_2(\gamma_1,\gamma_2,\gamma_3,\tau) - \gamma_1\gamma_3\tilde F_3(\gamma_1,\gamma_2,\gamma_3,\tau)]}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}\,d\tau \\
&\quad + e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)t}U_1^0(\gamma_1,\gamma_2,\gamma_3),
\end{aligned} \tag{3.29}$$

$$
\begin{aligned}
&U_2(\gamma_1,\gamma_2,\gamma_3,t)\\
&= \int_0^t e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)(t-\tau)}\\
&\quad\times \frac{[(\gamma_3^2+\gamma_1^2)\tilde F_2(\gamma_1,\gamma_2,\gamma_3,\tau)-\gamma_2\gamma_3\tilde F_3(\gamma_1,\gamma_2,\gamma_3,\tau)-\gamma_2\gamma_1\tilde F_1(\gamma_1,\gamma_2,\gamma_3,\tau)]}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}\,d\tau\\
&\quad+ e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)t}U_2^0(\gamma_1,\gamma_2,\gamma_3),
\end{aligned}
\tag{3.30}
$$

$$
\begin{aligned}
&U_3(\gamma_1,\gamma_2,\gamma_3,t)\\
&= \int_0^t e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)(t-\tau)}\\
&\quad\times \frac{[(\gamma_1^2+\gamma_2^2)\tilde F_3(\gamma_1,\gamma_2,\gamma_3,\tau)-\gamma_3\gamma_1\tilde F_1(\gamma_1,\gamma_2,\gamma_3,\tau)-\gamma_3\gamma_2\tilde F_2(\gamma_1,\gamma_2,\gamma_3,\tau)]}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}\,d\tau\\
&\quad+ e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)t}U_3^0(\gamma_1,\gamma_2,\gamma_3)\,.
\end{aligned}
\tag{3.31}
$$

Using the Fourier inversion formula (7.1) we obtain

$$
\begin{aligned}
&u_1(x_1,x_2,x_3,t)\\
&= \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\int_0^t e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)(t-\tau)}\frac{[(\gamma_2^2+\gamma_3^2)\tilde F_1(\gamma_1,\gamma_2,\gamma_3,\tau)]}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}\,d\tau\\
&\quad-\int_0^t e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)(t-\tau)}\frac{[\gamma_1\gamma_2\tilde F_2(\gamma_1,\gamma_2,\gamma_3,\tau)+\gamma_1\gamma_3\tilde F_3(\gamma_1,\gamma_2,\gamma_3,\tau)]}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}\,d\tau\\
&\quad+ e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)t}U_1^0(\gamma_1,\gamma_2,\gamma_3)\Big]e^{-i(x_1\gamma_1+x_2\gamma_2+x_3\gamma_3)}\,d\gamma_1d\gamma_2d\gamma_3\\
&= \frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(\gamma_2^2+\gamma_3^2)}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}\Big[\int_0^t e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)(t-\tau)}\\
&\quad\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(\tilde x_1\gamma_1+\tilde x_2\gamma_2+\tilde x_3\gamma_3)}\tilde f_1(\tilde x_1,\tilde x_2,\tilde x_3,\tau)\,d\tilde x_1d\tilde x_2d\tilde x_3d\tau\Big]\\
&\quad\times e^{-i(x_1\gamma_1+x_2\gamma_2+x_3\gamma_3)}\,d\gamma_1d\gamma_2d\gamma_3\\
&\quad-\frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\gamma_1\gamma_2}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}\Big[\int_0^t e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)(t-\tau)}\\
&\quad\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(\tilde x_1\gamma_1+\tilde x_2\gamma_2+\tilde x_3\gamma_3)}\tilde f_2(\tilde x_1,\tilde x_2,\tilde x_3,\tau)\,d\tilde x_1d\tilde x_2d\tilde x_3d\tau\Big]\\
&\quad\times e^{-i(x_1\gamma_1+x_2\gamma_2+x_3\gamma_3)}\,d\gamma_1d\gamma_2d\gamma_3\\
&\quad-\frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\gamma_1\gamma_3}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}\Big[\int_0^t e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)(t-\tau)}\\
&\quad\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(\tilde x_1\gamma_1+\tilde x_2\gamma_2+\tilde x_3\gamma_3)}\tilde f_3(\tilde x_1,\tilde x_2,\tilde x_3,\tau)\,d\tilde x_1d\tilde x_2d\tilde x_3d\tau\Big]\\
&\quad\times e^{-i(x_1\gamma_1+x_2\gamma_2+x_3\gamma_3)}\,d\gamma_1d\gamma_2d\gamma_3\\
&\quad+\frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)t}
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\tilde{x}_1\gamma_1+\tilde{x}_2\gamma_2+\tilde{x}_3\gamma_3)} u_1^0(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3)\,d\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_3\Big]\\
&\times e^{-i(x_1\gamma_1+x_2\gamma_2+x_3\gamma_3)}\,d\gamma_1 d\gamma_2 d\gamma_3\\
&= S_{11}(\tilde{f}_1)+S_{12}(\tilde{f}_2)+S_{13}(\tilde{f}_3)+B(u_1^0) \qquad (3.32)
\end{aligned}
$$

(see Remark 7.1);

$$
\begin{aligned}
&u_2(x_1,x_2,x_3,t)\\
&=\frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\int_0^t e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)(t-\tau)}\frac{[(\gamma_3^2+\gamma_1^2)\tilde{F}_2(\gamma_1,\gamma_2,\gamma_3,\tau)]}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}\,d\tau\\
&\quad-\int_0^t e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)(t-\tau)}\frac{[\gamma_2\gamma_3\tilde{F}_3(\gamma_1,\gamma_2,\gamma_3,\tau)+\gamma_2\gamma_1\tilde{F}_1(\gamma_1,\gamma_2,\gamma_3,\tau)]}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}\,d\tau\\
&\quad+e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)t}U_2^0(\gamma_1,\gamma_2,\gamma_3)\Big]e^{-i(x_1\gamma_1+x_2\gamma_2+x_3\gamma_3)}\,d\gamma_1 d\gamma_2 d\gamma_3\\
&=-\frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\gamma_2\gamma_1}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}\Big[\int_0^t e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)(t-\tau)}\\
&\quad\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\tilde{x}_1\gamma_1+\tilde{x}_2\gamma_2+\tilde{x}_3\gamma_3)}\tilde{f}_1(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tau)\,d\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_3 d\tau\Big]\\
&\quad\times e^{-i(x_1\gamma_1+x_2\gamma_2+x_3\gamma_3)}\,d\gamma_1 d\gamma_2 d\gamma_3\\
&\quad+\frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(\gamma_3^2+\gamma_1^2)}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}\Big[\int_0^t e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)(t-\tau)}\\
&\quad\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\tilde{x}_1\gamma_1+\tilde{x}_2\gamma_2+\tilde{x}_3\gamma_3)}\tilde{f}_2(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tau)\,d\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_3 d\tau\Big]\\
&\quad\times e^{-i(x_1\gamma_1+x_2\gamma_2+x_3\gamma_3)}\,d\gamma_1 d\gamma_2 d\gamma_3\\
&\quad-\frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\gamma_2\gamma_3}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}\Big[\int_0^t e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)(t-\tau)}\\
&\quad\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\tilde{x}_1\gamma_1+\tilde{x}_2\gamma_2+\tilde{x}_3\gamma_3)}\tilde{f}_3(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tau)\,d\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_3 d\tau\Big]\\
&\quad\times e^{-i(x_1\gamma_1+x_2\gamma_2+x_3\gamma_3)}\,d\gamma_1 d\gamma_2 d\gamma_3\\
&\quad+\frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)t}\\
&\quad\times\Big[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\tilde{x}_1\gamma_1+\tilde{x}_2\gamma_2+\tilde{x}_3\gamma_3)}u_2^0(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3)\,d\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_3\Big]\\
&\quad\times e^{-i(x_1\gamma_1+x_2\gamma_2+x_3\gamma_3)}\,d\gamma_1 d\gamma_2 d\gamma_3\\
&= S_{21}(\tilde{f}_1)+S_{22}(\tilde{f}_2)+S_{23}(\tilde{f}_3)+B(u_2^0), \qquad (3.33)
\end{aligned}
$$

(see Remark 7.1);

$$
\begin{aligned}
&u_3(x_1,x_2,x_3,t)\\
&=\frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\int_0^t e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)(t-\tau)}\frac{[(\gamma_1^2+\gamma_2^2)\tilde{F}_3(\gamma_1,\gamma_2,\gamma_3,\tau)]}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}\,d\tau\\
&\quad-\int_0^t e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)(t-\tau)}\frac{[\gamma_3\gamma_1\tilde{F}_1(\gamma_1,\gamma_2,\gamma_3,\tau)+\gamma_3\gamma_2\tilde{F}_2(\gamma_1,\gamma_2,\gamma_3,\tau)]}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)t} U_3^0(\gamma_1,\gamma_2,\gamma_3)\Big] e^{-i(x_1\gamma_1+x_2\gamma_2+x_3\gamma_3)}\, d\gamma_1 d\gamma_2 d\gamma_3 \\
&= -\frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\gamma_3\gamma_1}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}\Big[\int_0^t e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)(t-\tau)} \\
&\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\tilde{x}_1\gamma_1+\tilde{x}_2\gamma_2+\tilde{x}_3\gamma_3)} \tilde{f}_1(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tau)\, d\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_3 d\tau\Big] \\
&\times e^{-i(x_1\gamma_1+x_2\gamma_2+x_3\gamma_3)}\, d\gamma_1 d\gamma_2 d\gamma_3 \\
&- \frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\gamma_3\gamma_2}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}\Big[\int_0^t e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)(t-\tau)} \\
&\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\tilde{x}_1\gamma_1+\tilde{x}_2\gamma_2+\tilde{x}_3\gamma_3)} \tilde{f}_2(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tau)\, d\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_3 d\tau\Big] \\
&\times e^{-i(x_1\gamma_1+x_2\gamma_2+x_3\gamma_3)}\, d\gamma_1 d\gamma_2 d\gamma_3 \\
&+ \frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(\gamma_1^2+\gamma_2^2)}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}\Big[\int_0^t e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)(t-\tau)} \\
&\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\tilde{x}_1\gamma_1+\tilde{x}_2\gamma_2+\tilde{x}_3\gamma_3)} \tilde{f}_3(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tau)\, d\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_3 d\tau\Big] \\
&\times e^{-i(x_1\gamma_1+x_2\gamma_2+x_3\gamma_3)}\, d\gamma_1 d\gamma_2 d\gamma_3 \\
&+ \frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\nu(\gamma_1^2+\gamma_2^2+\gamma_3^2)t} \\
&\times \Big[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\tilde{x}_1\gamma_1+\tilde{x}_2\gamma_2+\tilde{x}_3\gamma_3)} u_3^0(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3)\, d\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_3\Big] \\
&\times e^{-i(x_1\gamma_1+x_2\gamma_2+x_3\gamma_3)}\, d\gamma_1 d\gamma_2 d\gamma_3 \\
&= S_{31}(\tilde{f}_1) + S_{32}(\tilde{f}_2) + S_{33}(\tilde{f}_3) + B(u_3^0),
\end{aligned} \tag{3.34}
$$

(see Remark 7.1).

Here $S_{11}()$, $S_{12}()$, $S_{13}()$, $S_{21}()$, $S_{22}()$, $S_{23}()$, $S_{31}()$, $S_{32}()$, $S_{33}()$, $B()$ are integral operators, and satisfy

$$
S_{12}() = S_{21}(), \quad S_{13}() = S_{31}(), \quad S_{23}() = S_{32}()\,.
$$

From the three expressions above for u_1, u_2, u_3, it follows that the vector $\vec{u}$ can be represented as:

$$
\vec{u} = \bar{\bar{S}} \cdot \vec{\tilde{f}} + \bar{\bar{B}} \cdot \vec{u}^0 = \bar{\bar{S}} \cdot \vec{f} - \bar{\bar{S}} \cdot (\vec{u}\cdot\nabla)\vec{u} + \bar{\bar{B}} \cdot \vec{u}^0\,, \tag{3.35}
$$

where $\vec{\tilde{f}}$ is determined by formula (2.14).

Here $\bar{\bar{S}}$ and $\bar{\bar{B}}$ are the matrix integral operators:

$$
\begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}, \quad \begin{pmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{pmatrix}.
$$

4. Spaces S and $\vec{TS}$

As in [7, 19], we consider the space S of all infinitely differentiable functions φ(x) defined in N-dimensional space $\mathbb{R}^N$ $(N = 3)$, such that these functions tend to 0 as $|x| \to \infty$, as well as their derivatives of any order, more rapidly than any power of $1/|x|$.

To define a topology in the space S let us introduce countable system of norms

$$\|\varphi\|_p = \sup_x \{|x^k D^q \varphi(x)|,\, |k| \le p, |q| \le p\} \quad p = 0, 1, 2, \dots, \tag{4.1}$$

where

$$|x^k D^q \varphi(x)| = |x_1^{k_1} \dots x_N^{k_N} \frac{\partial^{q_1 + \dots + q_N} \varphi(x)}{\partial x_1^{q_1} \dots \partial x_N^{q_N}}|,$$

$$k = (k_1, \dots, k_N), \quad q = (q_1, \dots, q_N), \quad x^k = x_1^{k_1} \dots x_N^{k_N},$$

$$D^q = \frac{\partial^{q_1 + \dots + q_N}}{\partial x_1^{q_1} \dots \partial x_N^{q_N}}, \quad q_1, \dots, q_N = 0, 1, 2, \dots .$$

Note that S is a perfect space (complete countably normed space, in which the bounded sets are compact). The space $\vec{TS}$ of vector-functions $\vec{\varphi}$ is a direct sum of N perfect spaces S $(N = 3)$ [26]:

$$\vec{TS} = S \oplus S \oplus S.$$

To define a topology in the space $\vec{TS}$ let us introduce countable system of norms

$$\|\vec{\varphi}\|_p = \sum_{i=1}^N \|\varphi_i\|_p = \sum_{i=1}^N \sup_x \{|x^k D^q \varphi_i(x)|,\, |k| \le p, |q| \le p\}, \tag{4.2}$$

$N = 3$, $p = 0, 1, 2, \dots$. The Fourier transform maps the space S onto the whole space S, and maps the space $\vec{TS}$ onto the whole space $\vec{TS}$ [23, 7].

5. Equivalence of the Cauchy problem in differential form (2.1)–(2.3) and in integral form

Let us denote solution of (2.1)–(2.3) as $\{\vec{u}(x_1, x_2, x_3, t), \mathrm{p}(x_1, x_2, x_3, \mathrm{t})\}$; in other words let us consider the infinitely differentiable by $t \in [0, \infty)$ vector-function $\vec{u}(x_1, x_2, x_3, t) \in \vec{TS}$, and infinitely differentiable function $p(x_1, x_2, x_3, t) \in S$, that turn equations (2.1) and (2.2) into identities. Vector-function $\vec{u}(x_1, x_2, x_3, t)$ also satisfies the initial condition (2.3) ($\vec{u}^0(x_1, x_2, x_3) \in \vec{TS}$):

$$\vec{u}(x_1, x_2, x_3, t)|_{t=0} = \vec{u}^0(x_1, x_2, x_3) \tag{5.1}$$

Let us put $\{\vec{u}(x_1, x_2, x_3, t), p(x_1, x_2, x_3, t)\}$ into equations (2.1), (2.2) and apply Fourier and Laplace transforms to the result identities considering initial condition (2.3). After all required operations (as in sections 2 and 3) we receive that vector-function $\vec{u}(x_1, x_2, x_3, t)$ satisfies integral equation

$$\vec{u} = \bar{\bar{S}} \cdot \vec{f} - \bar{\bar{S}} \cdot (\vec{u} \cdot \nabla)\vec{u} + \bar{\bar{B}} \cdot \vec{u}^0 \ = \bar{\bar{S}}^{\nabla} \cdot \vec{u} \tag{5.2}$$

Then the vector-function grad $p \in \vec{TS}$ is defined by equations (2.1) where vector-function $\vec{u}$ is defined by (5.2).

Here $\vec{f} \in \vec{TS}$, $\vec{u}^0 \in \vec{TS}$ and $\bar{\bar{S}}, \bar{\bar{B}}, \bar{\bar{S}}^{\nabla}$ are matrix integral operators. Vector-functions $\bar{\bar{S}} \cdot \vec{f}$, $\bar{\bar{B}} \cdot \vec{u}^0$, $\bar{\bar{S}} \cdot (\vec{u} \cdot \nabla)\vec{u}$ also belong $\vec{TS}$ since Fourier transform maps perfect space $\vec{TS}$ onto $\vec{TS}$.

Going from the other side, let us assume that $\vec{u}(x_1, x_2, x_3, t) \in \vec{TS}$ is continuous in $t \in [0, \infty)$ solution of integral equation (5.2). Integral-operators $S_{ij} \cdot (\vec{u} \cdot \nabla)\vec{u}$ are continuous in $t \in [0, \infty)$ [see (3.32)–(3.34)]. From here we obtain that according to (5.2),

$$\vec{u}(x_1, x_2, x_3, 0) = \vec{u}^0(x_1, x_2, x_3)$$

also that $\vec{u}(x_1,x_2,x_3,t)$ is differentiable by $t \in [0,\infty)$. As described before, the Fourier transform maps the perfect space $\vec{TS}$ on itself. Hence, $\{\vec{u}(x_1,x_2,x_3,t)$ and $p(x_1,x_2,x_3,t)\}$ is the solution of the Cauchy problem (2.1)–(2.3). From here we see that solving the Cauchy problem (2.1)–(2.3) is equivalent to finding continuous in $t \in [0,\infty)$ solution of integral equation (5.2).

6. The Caccioppoli-Banach fixed point principle

See [10, 26, 20, 11, 8, 1]. Further we have $\vec{f} \equiv 0$. Let us rewrite integral equation (5.2) with this condition as

$$\vec{u} = -\bar{\bar{S}} \cdot (\vec{u} \cdot \nabla)\vec{u} + \bar{\bar{B}} \cdot \vec{u}^0 \tag{6.1}$$

Let us divide all parts of the integral equation (6.1) by some constant V, that we will define appropriately below. Then we receive modified integral equation equivalent to equation (6.1):

$$\vec{u}_V = -\bar{\bar{S}}_V \cdot (\vec{u}_V \cdot \nabla_V)\vec{u}_V + \bar{\bar{B}}_V \cdot \vec{u}^0_V = \bar{\bar{S}}^{\nabla}_V(\vec{u}_V) \tag{6.2}$$

here

$$\begin{gathered} \vec{u}_V = \frac{\vec{u}}{V}, \quad \vec{u}^0_V = \frac{\vec{u}^0}{V}, \quad \nabla_V = V \cdot \nabla, \\ x_{kV} = \frac{x_k}{V}, \quad \gamma_{kV} = V \cdot \gamma_k, \quad \nu_V = \frac{\nu}{V^2}, \quad 1 \le k \le N\,. \end{gathered} \tag{6.3}$$

The selection of the constant V is based on a priori estimate of the solution of the Cauchy problem for the Navier-Stokes equations [13, 12]:

$$\|\vec{u}\|_{L_2} \le \|\vec{u^0}\|_{L_2} \tag{6.4}$$

where

$$\|\vec{u}\|_{L_2} = \Big(\int_{\mathbb{R}^3} |\vec{u}|^2 dx\Big)^{1/2}.$$

The space of vector-functions $\vec{TS}$ is divided into two subsets. First subset consists of the vector-functions “rapidly rapidly”decreasing to zero for $|x| \to \infty$. This subset is defined by the inequality

$$\|\vec{u}\|_{L_2} \le B\|\vec{u}\|_{C^2}\,, \tag{6.5}$$

where B is a fixed constant. Second subset consists of the vector-functions “rapidly” decreasing to zero for $|x| \to \infty$. This subset is defined by the inequality

$$\|\vec{u}\|_{C^2} \le B'\|\vec{u}\|_{L_2}, \tag{6.6}$$

where B' is a fixed constant.

$$\|\vec{u}\|_{C^2} = \sum_{i=1}^{N} \|u_i\|_{C^2} = \sum_{i=1}^{N}\sum_{q=0}^{2} \max_x |D^q u_i(x)|, \quad N = 3.$$

For the first subset by using appropriate sequence of inequalities [7], (6.4), (6.5),

$$\|\vec{u}\|_{L_2} \le B\|\vec{u}\|_{C^2} \le K\|\vec{u}\|_{L_2} \le K\|\vec{u^0}\|_{L_2} \le KB\|\vec{u^0}\|_{C^2} \tag{6.7}$$

we obtain

$$\|\vec{u}\|_{C^2} \le K\|\vec{u^0}\|_{C^2}, \tag{6.8}$$

where K is a constant.

For the second subset by using appropriate sequence of inequalities [7], (6.4), (6.6),

$$\|\vec{u}\|_{C^2} \le B'\|\vec{u}\|_{L_2} \le B'\|\vec{u^0}\|_{L_2} \tag{6.9}$$

we obtain

$$\|\vec{u}\|_{C^2} \le B' \cdot \|\vec{u^0}\|_{L_2}\,. \tag{6.10}$$

Let us choose the constant V as:

$$\max\{K\|\vec{u^0}\|_{C^2},\ B'\|\vec{u^0}\|_{L_2}\} < V \tag{6.11}$$

Then we have

$$\|\vec{u}_V\|_{C^2} < 1 \tag{6.12}$$

We can use the fixed point principle to prove existence and uniqueness of the solution of integral equation (6.2).

For this purpose we will operate with the following properties of matrix integral operator $\bar{\bar{S}}_V^{\nabla}$:

1. The matrix integral operator $\bar{\bar{S}}_V^{\nabla}$ depends continuously on its parameter $t \in [0,\infty)$ (based on formulas (3.32)–(3.34)).

2. The matrix integral operator $\bar{\bar{S}}_V^{\nabla}$ maps vector-functions $\vec{u}_V$ from perfect space $\vec{TS}$ onto perfect space $\vec{TS}$. This property directly follows from the properties of Fourier transform [7], and the form of integrands of integral operators S_{ij}, B (based on formulas (3.32)–(3.34)). Hence in this case we can consider the convergence of functions from $\vec{TS}$ not only in countable system of norms (4.2), but also in norm C^2.

3. Matrix integral operator $\bar{\bar{S}}_V$ is "quadratic".

4. $\|\bar{\bar{S}}_V^{\nabla} \cdot \vec{u}_V - \bar{\bar{S}}_V^{\nabla} \cdot \vec{u}'_V\|_{C^2} < \|\vec{u}_V - \vec{u}'_V\|_{C^2}$ for any $\vec{u}_V,\ \vec{u}'_V \in \vec{TS}$ ($\vec{u}_V \neq \vec{u}'_V$) and any $t \in [0,\infty)$ (based on properties 1, 2, 3 and formulas (3.32)–(3.34), (6.12)). The properties mentioned above allow us to prove that matrix integral operator $\bar{\bar{S}}_V^{\nabla}$ is a contraction operator.

Theorem 6.1 ([10]). *The matrix integral operator $\bar{\bar{S}}_V^{\nabla}$ maps the perfect space $\vec{TS}$ onto the perfect space $\vec{TS}$, and for any $\vec{u}_V, \vec{u}'_V \in \vec{TS}$ ($\vec{u}_V \neq \vec{u}'_V$) the condition 4 is valid. Then the matrix integral operator $\bar{\bar{S}}_V^{\nabla}$ is a contraction operator; i.e., the following condition is true*

$$\|\bar{\bar{S}}_V^{\nabla}(\vec{u}_V) - \bar{\bar{S}}_V^{\nabla}(\vec{u}'_V)\|_{C^2} \le \alpha \cdot \|\vec{u}_V - \vec{u}'_V\|_{C^2} \tag{6.13}$$

where $\alpha < 1$ and is independent from $\vec{u}_V,\ \vec{u}'_V \in \vec{TS}$ for any $t \in [0,\infty)$.

Proof. By contradiction, let us assume that the opposite is true. Then there exist such $\vec{u}_{Vn}, \vec{u}'_{Vn} \in \vec{TS}$ ($n = 1,2,\dots$) and

$$\lim_{n\to\infty} \vec{u}_{Vn}, \lim_{n\to\infty} \vec{u}'_{Vn} \in \vec{TS}$$

that

$$\|\bar{\bar{S}}_V^{\nabla}(\vec{u}_{Vn}) - \bar{\bar{S}}_V^{\nabla}(\vec{u}'_{Vn})\|_{C^2} = \alpha_n \cdot \|\vec{u}_{Vn} - \vec{u}'_{Vn}\|_{C^2} \quad n = 1,2,\dots;\ \alpha_n \to 1 \tag{6.14}$$

Then the limiting result in (6.14) would lead to equality

$$\|\bar{\bar{S}}_V^{\nabla}(\vec{u}_V) - \bar{\bar{S}}_V^{\nabla}(\vec{u}'_V)\|_{C^2} = \|\vec{u}_V - \vec{u}'_V\|_{C^2},$$

which contradicts condition 4. Hence, $\bar{\bar{S}}_V^{\nabla}$ is a contraction operator. □

Next, we have the existence and uniqueness of a solution [10].

Theorem 6.2. *Let us consider a contraction operator $\bar{\bar{S}}_V^{\nabla}$. Then there exists the unique solution $\vec{u}_V^*$ of equation (6.2) in space $\vec{TS}$ for any $t \in [0,\infty)$. Also in this case it is possible to obtain $\vec{u}_V^*$ as a limit of sequence $\{\vec{u}_{Vn}\}$, where*

$$\vec{u}_{V,n+1} = \bar{\bar{S}}_V^{\nabla}(\vec{u}_{Vn}) \quad n = 0, 1, \dots,$$

and $\vec{u}_{V0} = 0$.

The rate of conversion of the sequence $\{\vec{u}_{Vn}\}$ to the solution can be defined from the inequality

$$\|\vec{u}_{Vn} - \vec{u}_V^*\|_{C^2} \le \frac{\alpha^n}{(1-\alpha)} \|\vec{u}_{V1} - \vec{u}_{V0}\|_{C^2} \quad n = 0, 1, \dots . \tag{6.15}$$

Proof. It is clear that

$$\vec{u}_{V,n+1} = \bar{\bar{S}}_V^{\nabla}(\vec{u}_{Vn}), \quad \vec{u}_{Vn} = \bar{\bar{S}}_V^{\nabla}(\vec{u}_{V,n-1}).$$

It follows from (6.13) that

$$\|\vec{u}_{V,n+1} - \vec{u}_{Vn}\|_{C^2} \le \alpha \cdot \|\vec{u}_{Vn} - \vec{u}_{V,n-1}\|_{C^2}.$$

Using similar inequalities one after another while decreasing n we will obtain

$$\|\vec{u}_{V,n+1} - \vec{u}_{Vn}\|_{C^2} \le \alpha^n \cdot \|\vec{u}_{V1} - \vec{u}_{V0}\|_{C^2}.$$

From this result it follows that

$$\begin{aligned}
\|\vec{u}_{V,n+l} - \vec{u}_{Vn}\|_{C^2} &\le \|\vec{u}_{V,n+l} - \vec{u}_{V,n+l-1}\|_{C^2} + \dots + \|\vec{u}_{V,n+1} - \vec{u}_{Vn}\|_{C^2} \\
&\le (\alpha^{n+l-1} + \dots + \alpha^n)\|\vec{u}_{V1} - \vec{u}_{V0}\|_{C^2} \\
&\le \frac{\alpha^n}{(1-\alpha)} \|\vec{u}_{V1} - \vec{u}_{V0}\|_{C^2}.
\end{aligned} \tag{6.16}$$

Because of $\alpha^n \to 0$ for $n \to \infty$, the obtained estimate (6.16) shows that sequence $\{\vec{u}_{Vn}\}$ is a Cauchy sequence. Since the space $\vec{TS}$ is a perfect space, this sequence converges to an element $\vec{u}_V^* \in \vec{TS}$, such that $\bar{\bar{S}}_V^{\nabla}(\vec{u}_V^*)$ has sense. We use inequality (6.13) again and have:

$$\|\vec{u}_{V,n+1} - \bar{\bar{S}}_V^{\nabla}(\vec{u}_V^*)\|_{C^2} = \|\bar{\bar{S}}_V^{\nabla}(\vec{u}_{Vn}) - \bar{\bar{S}}_V^{\nabla}(\vec{u}_V^*)\|_{C^2} \le \alpha \|\vec{u}_{Vn} - \vec{u}_V^*\|_{C^2} \quad n = 0, 1, 2, \dots$$

The right part of the above inequality tends to 0 as $n \to \infty$ and it means that $\vec{u}_{V,n+1} \to \bar{\bar{S}}_V^{\nabla}(\vec{u}_V^*)$ and $\vec{u}_V^* = \bar{\bar{S}}_V^{\nabla}(\vec{u}_V^*)$. In other words, $\vec{u}_V^*$ is the solution of equation (6.2).

Uniqueness of the solution also follows from (6.13). In fact, if there would exist another solution $\widetilde{\vec{u}_V} \in \vec{TS}$, then

$$\|\widetilde{\vec{u}_V} - \vec{u}_V^*\|_{C^2} = \|\bar{\bar{S}}_V^{\nabla}(\widetilde{\vec{u}_V}) - \bar{\bar{S}}_V^{\nabla}(\vec{u}_V^*)\|_{C^2} \le \alpha \cdot \|\widetilde{\vec{u}_V} - \vec{u}_V^*\|_{C^2}.$$

Such situation could happen only if $\|\widetilde{\vec{u}_V} - \vec{u}_V^*\|_{C^2} = 0$, or $\widetilde{\vec{u}_V} = \vec{u}_V^*$.

We can also obtain the estimate (6.15) from estimate (6.16) as a limiting result as $l \to \infty$.

Now let us show that continuous dependence of operator $\bar{\bar{S}}_V^{\nabla}$ on t leads to continuous dependence of the solution of the problem on t.

We will say that matrix integral operator $\bar{\bar{S}}_V^{\nabla}$ is continuous in t at a point $t_0 \in [0,\infty)$, if for any sequence $\{t_n\} \in [0,\infty)$ with $t_n \to t_0$ for $n \to \infty$, the following is true:

$$\bar{\bar{S}}_{Vt_n}^{\nabla}(\vec{u}_V) \to \bar{\bar{S}}_{Vt_0}^{\nabla}(\vec{u}_V) \quad \text{for any } \vec{u}_V \in \vec{TS}. \tag{6.17}$$

$\square$

From Theorem 6.2 it follows that for any $t \in [0, \infty)$, equation (6.2) has the unique solution, which depends on t. Let us denote it as $\vec{u}^*_{Vt}$. We will say that solution of equation (6.2) depends continuously on t at $t = t_0$, if for any sequence $\{t_n\} \in [0\infty)$ with $t_n \to t_0$ for $n \to \infty$, the following is true:

$$\vec{u}^*_{Vt_n} \to \vec{u}^*_{Vt_0}.$$

Next, we have the continuous dependence of the solution on t.

Theorem 6.3 ([10]). *Let us consider operator $\bar{\bar{S}}^{\nabla}_{Vt}$ that satisfies condition* (6.13) *for any $t \in [0, \infty)$, where α is independent from t and that operator $\bar{\bar{S}}^{\nabla}_{Vt}$ is continuous in t at a point $t_0 \in [0, \infty)$. Then for $t = t_0$ the solution of* (6.2) *depends continuously on t.*

Proof. Let us consider any $t \in [0, \infty)$. We will construct the solution $\vec{u}^*_{Vt}$ of equation (6.2) as a limit of sequence $\{\vec{u}_{Vn}\}$:

$$\vec{u}_{V,n+1} = \bar{\bar{S}}^{\nabla}_{Vt}(\vec{u}_{Vn}) \quad n = 0, 1, \dots;\ \vec{u}_{V0} = \vec{u}^*_{Vt_0} \tag{6.18}$$

Let us rewrite inequality (6.15) for n = 0:

$$\|\vec{u}^*_V - \vec{u}_{V0}\|_{C^2} \le \frac{1}{(1-\alpha)} \|\vec{u}_{V1} - \vec{u}_{V0}\|_{C^2} \tag{6.19}$$

Since $\vec{u}^*_{Vt_0} = \bar{\bar{S}}^{\nabla}_{Vt_0}(\vec{u}^*_{Vt_0})$, because of (6.18) and (6.19) we have

$$\|\vec{u}^*_{Vt} - \vec{u}^*_{Vt_0}\|_{C^2} \le \frac{1}{(1-\alpha)} \|\vec{u}_{V1} - \vec{u}_{V0}\|_{C^2} = \frac{1}{(1-\alpha)} \|\bar{\bar{S}}^{\nabla}_{Vt}(\vec{u}^*_{Vt_0}) - \bar{\bar{S}}^{\nabla}_{Vt_0}(\vec{u}^*_{Vt_0})\|_{C^2} \tag{6.20}$$

Now with the help of (6.17) we obtain the required continuity of $\vec{u}_{Vt}$ for $t = t_0$. □

Following (6.3) and (6.11) we obtain the result:

$$\vec{u} = \vec{u}_V \cdot V, \quad \nu = \nu_V \cdot V^2. \tag{6.21}$$

Then vector-function $\nabla p \in \vec{TS}$ is defined by (2.1) where vector-function $\vec{u}$ is received from equation (6.21). Function p is defined up to an arbitrary constant.

Remark 6.4. From the above statements, it follows that there exists the unique set of smooth functions $u_{\infty i}(x, t)$, $p_\infty(x, t)$ $(i = 1, 2, 3)$ $\mathbb{R}^3 \times [0, \infty)$ that satisfies (2.1), (2.2), (2.3) and

$$u_{\infty i}, p_\infty \in C^\infty(\mathbb{R}^3 \times [0, \infty)), \tag{6.22}$$

Then, using the inequality $\|\vec{u}\|_{L_2} \le \|\vec{u^0}\|_{L_2}$ from (6.4), [13], [12], we have

$$\int_{\mathbb{R}^3} |\vec{u}_\infty(x, t)|^2 dx < C, \quad \forall t \ge 0. \tag{6.23}$$

Let us consider $\nu \to 0$ in integral operator $\bar{\bar{S}}^{\nabla}_V$. Then we see that Theorems 6.1–6.3 are correct also in case of Euler equations; i.e., there exists unique smooth solution in all time range for this case.

7. Appendix

The Fourier integral can be stated in the forms:

$$\begin{aligned} U(\gamma_1,\gamma_2,\gamma_3) &= F[u(x_1,x_2,x_3)] \\ &= \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x_1,x_2,x_3)e^{i(\gamma_1x_1+\gamma_2x_2+\gamma_3x_3)}dx_1dx_2dx_3 \\ u(x_1,x_2,x_3) &= \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(\gamma_1,\gamma_2,\gamma_3)\, e^{-i(\gamma_1x_1+\gamma_2x_2+\gamma_3x_3)}d\gamma_1d\gamma_2d\gamma_3 \end{aligned} \tag{7.1}$$

The Laplace integral is usually stated in the form

$$U^{\otimes}(\eta) = L[u(t)] = \int_0^{\infty} u(t)e^{-\eta t}dt \quad u(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} U^{\otimes}(\eta)e^{\eta t}d\eta \quad c > c_0. \tag{7.2}$$

Then

$$L[u'(t)] = \eta U^{\otimes}(\eta) - u(0). \tag{7.3}$$

The convolution theorem [6, 30] is stated as: If integrals

$$U_1^{\otimes}(\eta) = \int_0^{\infty} u_1(t)e^{-\eta t}d\,t \quad U_2^{\otimes}(\eta) = \int_0^{\infty} u_2(t)e^{-\eta t}d\,t$$

converge absolutely for $\operatorname{Re}\eta > \sigma_d$, then $U^{\otimes}(\eta) = U_1^{\otimes}(\eta)U_2^{\otimes}(\eta)$ is Laplace transform of

$$u(t) = \int_0^t u_1(t-\tau)\,u_2(\tau)\,d\,\tau \tag{7.4}$$

A useful Laplace integral is

$$L[e^{\eta_k t}] = \int_0^{\infty} e^{-(\eta-\eta_k)\,t}d\,t = \frac{1}{(\eta-\eta_k)} \quad \operatorname{Re}\eta > \eta_k \tag{7.5}$$

Remark 7.1. In the calculations of integrals (3.32)–(3.34) for components of velocity u_1, u_2, u_3 for the inverse Fourier transforms, we have each integrand $\tilde{f}(\gamma_1,\gamma_2,\gamma_3)$ as a product of functions $\chi(\gamma_1,\gamma_2,\gamma_3)$ and $\varphi(\gamma_1,\gamma_2,\gamma_3)$,

$$\tilde{f}(\gamma_1,\gamma_2,\gamma_3) = \chi(\gamma_1,\gamma_2,\gamma_3)\cdot\varphi(\gamma_1,\gamma_2,\gamma_3),$$

where $\varphi(\gamma_1,\gamma_2,\gamma_3)$ belongs to space S (functions of $\gamma_1,\gamma_2,\gamma_3$) [7] and $\chi(\gamma_1,\gamma_2,\gamma_3)$ is one of the fractions:

$$\begin{gathered} \frac{(\gamma_2^2+\gamma_3^2)}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}, \quad \frac{(\gamma_1\cdot\gamma_2)}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}, \quad \frac{(\gamma_1\cdot\gamma_3)}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}, \\ \frac{(\gamma_3^2+\gamma_1^2)}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}, \quad \frac{(\gamma_2\cdot\gamma_3)}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)}, \quad \frac{(\gamma_1^2+\gamma_2^2)}{(\gamma_1^2+\gamma_2^2+\gamma_3^2)} \end{gathered}$$

These fractions are infinitely differentiable functions for $\gamma_1 \neq 0$, $\gamma_2 \neq 0$, $\gamma_3 \neq 0$ with one point of discontinuity $\gamma_1 = 0$, $\gamma_2 = 0$, $\gamma_3 = 0$ (The discontinuities have finite values at this point).

For these calculations the inverse Fourier transforms are defined as Lebesgue integrals with Cauchy principal values.

Theorem 7.2. *: The inverse Fourier transform of* $\tilde{f}(\gamma_1,\gamma_2,\gamma_3) = \chi(\gamma_1,\gamma_2,\gamma_3)\cdot\varphi(\gamma_1,\gamma_2,\gamma_3)$,

$$F[\tilde{f}] \equiv \psi(\sigma) \equiv \int_{-\infty}^{\infty} e^{i(\gamma,\sigma)}\chi(\gamma)\cdot\varphi(\gamma)d\gamma \tag{7.6}$$

as function of σ belongs to space S (functions of σ); i.e., $\psi(\sigma)$ has two criteria:

(1) *$\psi(\sigma)$ is infinitely differentiable function,*
(2) *when $|\sigma| \to \infty$, $\psi(\sigma)$ tends to 0, as well as its derivatives of any order, more rapidly than any power of $1/|\sigma|$.*

Integral in (7.6) admits of differentiation with respect to the parameter σ_j, since the integral obtained after formal differentiation remains absolutely convergent:

$$\frac{\partial \psi(\sigma)}{\partial \sigma_j} \equiv \int_{-\infty}^{\infty} i\gamma_j e^{i(\gamma,\sigma)} \chi(\gamma)\varphi(\gamma) d\gamma \tag{7.7}$$

The properties of function $\varphi(\gamma)$ permit this differentiation to be continued without limit. This means that the function $\psi(\sigma)$ is infinitely differentiable (see criterion 1).

To prove criterion 2 we create a function with parameter n:

$$f_n(\gamma_1,\gamma_2,\gamma_3) = ne^{-\frac{1}{n^2(\gamma_1^2+\gamma_2^2+\gamma_3^2)}} \chi(\gamma_1,\gamma_2,\gamma_3)\cdot\varphi(\gamma_1,\gamma_2,\gamma_3)$$

$n = 1, 2, 3 \dots.$ Then $f_n(\gamma_1,\gamma_2,\gamma_3)$ belongs to space S (functions of $\gamma_1,\gamma_2,\gamma_3$). Then we estimate the integral in formula (7.6) using the inverse Fourier transform of $f_n(\gamma)$ for $|\sigma| >> 0$, $|\sigma| \to \infty$:

$$|\psi(\sigma)| \equiv \Big|\int_{-\infty}^{\infty} e^{i(\gamma,\sigma)}\chi(\gamma)\varphi(\gamma)d\gamma\Big| \le \Big|n\int_{-\infty}^{\infty} e^{i(\gamma,\sigma)} e^{-\frac{1}{n^2\gamma^2}}\chi(\gamma)\varphi(\gamma)d\gamma\Big|, \tag{7.8}$$

$n >> 1, \dots < \infty$. Since $f_n(\gamma_1,\gamma_2,\gamma_3)$ belongs to space S (functions of $\gamma_1,\gamma_2,\gamma_3$) then the inverse Fourier transform of $f_n(\gamma_1,\gamma_2,\gamma_3)$ belongs to space S (functions of $\sigma_1,\sigma_2,\sigma_3$). Hence $F[f_n]$ is the function, such that when $|\sigma| \to \infty$ this function tends to 0, as well as its derivatives of any order, more rapidly than any power of $1/|\sigma|$.

So from formula (7.8), we have that $\psi(\sigma)$ is a function, such that when $|\sigma| \to \infty$ this function tends to 0, more rapidly than any power of $1/|\sigma|$.

Then we estimate the integral in formula (7.7). Using formula like (7.8) for $\frac{\partial\psi(\sigma)}{\partial\sigma_j}$, we obtain that $\frac{\partial\psi(\sigma)}{\partial\sigma_j}$ tends to 0 more rapidly than any power of $1/|\sigma|$. The properties of function $\varphi(\gamma)$ permit this differentiation to be continued without limit and further using of formulas like (7.8) to derivatives leads to the conclusion that all derivatives of function $\psi(\sigma)$ tend to 0 more rapidly than any power of $1/|\sigma|$.

We have proved that $\psi(\sigma)$ belongs to space S (functions of σ).

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Addendum posted by the editor on December 31, 2013

Two anonymous readers informed us that the results in this article are incorrect. I sent the questions to the authors who answer them, and posted a new version of their article in arXiv in September 2013; see reference [2] below. However, the answers seem to be still erroneous. Here is an extract of our correspondence.

Question 1. The proof of Point 4 page 12 (which is the core of the paper) does not make sense. I doubt that in the book by Gelfand one find the "appropriate sequence of inequalities" allowing to control a C^2 norm by a L^2 norm.
Answer 1. To improve the estimate of the constant V we have applied an approach that avoids the sequence of inequalities. Let us choose the constant V as:

$$\max\{\|\vec{u}^0\|_{C^2}, \|\vec{u}^0\|_{L_2}, (\|\vec{u}\|_{C_2} + \|\vec{u}\|_{L_2})\} \ll V$$

Then we have

$$|\vec{u}_V| < 1, \quad |\vec{u}_V^0| < 1$$

In property 4 of the matrix integral operator $\bar{\bar{S}}_V^{\nabla}$, and in Theorems 6.1–6.3, where two vector-functions $\vec{u}_V$ and $\vec{u}'_V$ are presented, constant V is selected as maximum between $V(\vec{u})$ and $V'(\vec{u}')$. In other words we do not use "appropriate sequence of inequalities" in the new version, see reference [2] below.
Reader's reply: As V is constant and as $\vec{u}$ is time-varying, does the inequality holds on $[0;1)$? Moreover, the role of V is still more dubious. Indeed, the aim seems to ensure that $\vec{u}$ is bounded by 1. But the role of this boundedness is unclear. It seems that the author want to use the fact that S is a perfect set (bounded subsets are relatively compact); but boundedness in S means being bounded for a whole countable family of semi-norms involving polynomial weights and higher derivatives, so that the mere boundedness of the modulus of $\vec{u}$ does not bear any compactness property in S.

Question 2. The proof of Theorem 6.1 is false, since u could be equal to u', leading to no contradiction with point 4.
Answer 2. Below you can see the condition ($\vec{u}_V \neq \vec{u}'_V$) in Theorem 6.1.

The matrix integral operator $\bar{\bar{S}}_V^{\nabla}$ maps the perfect space $\vec{TS}$ onto the perfect space $\vec{TS}$, and for any $\vec{u}_V, \vec{u}'_V \in \vec{TS}$ ($\vec{u}_V \neq \vec{u}'_V$) the condition 4 is valid. Then the matrix integral operator $\bar{\bar{S}}_V^{\nabla}$ is a contraction operator, i.e. the following condition is true:

$$\|\bar{\bar{S}}_V^{\nabla} \cdot \vec{u}_V - \bar{\bar{S}}_V^{\nabla} \cdot \vec{u}'_V\|_p \leq \alpha \cdot \|\vec{u}_V - \vec{u}'_V\|_p \tag{6.13}$$

where $\alpha < 1$ and is independent from $\vec{u}_V, \vec{u}'_V \in \vec{TS}$ for any t $\in [0, \infty)$.
Reader's reply: The condition $\vec{u} \neq \vec{u}'$ is stated in the assumption of Theorem 6.1. However, the proof uses two limit points of sequences $\vec{u}_n$ and $\vec{u}'_n$ with $\vec{u} \neq \vec{u}'$ and those two limit points may satisfy $\vec{u} = \vec{u}'$ and thus the alleged contradiction is not proved. As a matter of fact, there are a lot of examples of contraction on compacts spaces that are not strictly contractive: for instance $t \mapsto \sin t$ on $[0, \pi/2]$: we have $|\sin t - \sin t'| < |t - t'|$ when $t \neq t'$, however there is no $\alpha < 1$ such that $|\sin t - \sin t'| \leq \alpha|t - t'|$ for all $t, t' \in [0, \pi/2]$ (as the derivative of $\sin t$ at $t = 0$ equals 1).

Question 3. The proof of Theorem 6.2 is false: convergence in C^2 norm of a sequence of Schwartz functions does not imply that the limit is a Schwartz function
Answer 3. We use $\|\cdot\|_p$ countable system of norms (4.2) for the property 4 of the matrix integral operator $\bar{\bar{S}}_V^{\nabla}$, and in Theorems 6.1-6.3 now. The limit of a sequence of Schwartz functions is also a Schwartz function by $\|\cdot\|_p$-countable system of norms (4.2).

Reader's reply: The convergence in C^2 norm has been replaced by convergence in the countable family of norms that define the topology of S. This is better than in the published version, except that the convergence in those norms is not proved. The new Theorem 6.1 is false: the assertion that $\|S_V^\nabla \vec{u}_V - S_V^\nabla \vec{u}'_V\|_p < \|\vec{u}_V - \vec{u}'_V\|_p$ (condition 4) follows from formulas (3.30)-(3.31)-(3.32) and (6.5) is purely an act of faith, but not a plausible proof. There is absolutely no scientific reason that this infinite family of inequalities holds for a providential choice of V.

Question 4. Remark 6.4 is strange. How can we deal the case of the vanishing viscosity?
Answer 4. Let us consider $\nu \to 0$. We see that integral operator $\bar{\bar{S}}_V^\nabla$ and Theorems 6.1-6.3 are correct also in case of Euler equations, i.e. there exists unique smooth solution in all time range for this case too.
Reader's reply: The answer that "we see that theorems 6.1, 6.2 and 6.3 are correct in case of Euler equations" without any further explanations on how we can see it is meaningless.

Question 5. Theorem 7.2 is incorrect, as a discontinuous function cannot belong to the Schwartz class
Answer 5. $\varphi(\gamma)$ as function of γ belongs to space S (functions of γ). I.e., $\varphi(\gamma)$ has two properties:
(1) $\varphi(\gamma)$ is infinitely differentiable function,
(2) when $|\gamma| \to \infty$, $\varphi(\gamma)$ tends to 0, as well as its derivatives of any order, more rapidly than any power of $1/|\gamma|$. Function

$$\tilde{f}(\gamma_1, \gamma_2, \gamma_3) = \chi(\gamma_1, \gamma_2, \gamma_3) \cdot \varphi(\gamma_1, \gamma_2, \gamma_3),$$

does not belong to space S (functions of γ) then

$$\chi(\gamma_1, \gamma_2, \gamma_3)$$

is the discontinuous function for $\gamma = 0$. But the inverse Fourier transform of function $\tilde{f}(\gamma_1, \gamma_2, \gamma_3)$

$$F[\tilde{f}] \equiv \psi(\sigma) \equiv \int_{-\infty}^{\infty} e^{i(\gamma,\sigma)} \chi(\gamma) \cdot \varphi(\gamma) d\gamma$$

$\psi(\sigma)$ as function of σ belongs to space S (functions of σ). I.e., $\psi(\sigma)$ has two properties:
(1) $\psi(\sigma)$ is infinitely differentiable function,
(2) when $|\sigma| \to \infty$ $\psi(\sigma)$ tends to 0, as well as its derivatives of any order, more rapidly than any power of $1/|\sigma|$. We can see the correct proof of two properties (1) and (2) for $\psi(\sigma)$ in Theorem 7.2.
Reader's reply: Theorem 7.2 is incorrect. It is a basic fact of the theory of distributions that the Fourier transform is a bijection of the Schwartz class onto itself, so that obviously a non-smooth function cannot have an inverse Fourier transform in the Schwartz class. The author still argues that his proof is correct. The proof is based on the inequality

$$\Big| \int e^{i(\gamma,\sigma)} \chi(\gamma) \, d\gamma \Big| \leq n \Big| \int e^{i(\gamma,\sigma)} e^{-1/(n^2\gamma^2)} \chi(\gamma) \varphi(\gamma) \, d\gamma \Big|$$

which is not proved (and false). As a matter of fact, the inequality is valid for large n when σ is fixed, but is false for large σ when n is fixed.

Question 6. What are the main properties of the solution?

Answer 6. Solution $\vec{u} \in \vec{TS}$ and $p \in S$ In other words there exists the unique set of smooth functions $u_{\infty i}(x,t)$, $p_\infty(x,t)$ $(i = 1, 2, 3)$ on $\mathbb{R}^3 \times [0, \infty)$ that satisfies (2.1), (2.2), (2.3), and

$$u_{\infty i}, p_\infty \in C^\infty(\mathbb{R}^3 \times [0, \infty)),$$

Then, using the inequality $\|\vec{u}\|_{L_2} \le \|\vec{u}^0\|_{L_2}$, we have

$$\int_{\mathbb{R}^3} |\vec{u}_\infty(x,t)|^2 dx < C$$

for all $t \ge 0$, see Fefferman [1], below.

Question 7. I think that (6.7) and (6.9) are wrong (L^2 never controls C^2 with universal constants). After that, the authors get (6.8). From there using Beale-Kato-Majda they may conclude the global existence. Without (6.7) and (6.9), their definition of V makes no sense, as the new V will depend on time.

Answer 7. Instead inequalities (6.5)–(6.12) we choose the constant V as:

$$\max\{\|\vec{u}^0\|_{C^2}, \|\vec{u}^0\|_{L_2}, (\|\vec{u}\|_{C_2} + \|\vec{u}\|_{L_2})\} \ll V$$

Then we have that $|\vec{u}_V| < 1$, $|\vec{u}^0_V| < 1$. (Please see Answer 1.) V is a constant, and hence is not dependent on time.

Reader's reply: I do not understand how V can be constant and control the varying values of $\vec{u}$.

Thus, the revised version is still totally incorrect. As a matter of fact, it is well known that generically one cannot have a better decay than $O(|x|^{-4})$ for the solutions, so that any attempt to prove a fixed-point theorem in the Schwartz class for the Navier-Stokes equations can be but a disastrous failure.

Additional references:

[1] C. L. Fefferman; *Existence and smoothness of the Novier-Stokes equation*, The Clay Mathematics Institute, Official Problem Description.

[2] A. Tsionskiy, M. Tsionskiy; *Existence, uniqueness and smoothness of solution for 3D Navier-Stokes equations with any smooth initial velocity*, arXiv:1201.1609v8, September 1, 2013.

Arkadiy Tsionskiy, Mikhail Tsionskiy
Department of Civil, Environmental and Ocean Engineering, Stevens Institute of Technology, Hoboken, NJ 07030, USA
E-mail address: amtsionsk@gmail.com